Discrete Stochastic Models for Finance

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Foreword

These notes have been prepared on the occasion of the authors visit to the Mathematics Department of the University of the Philippines, in Manila-Diliman, in the framework of the Asia Link program of the European Community. Their purpose is to help all our colleagues here to offer to Filipino students attractive teachings onn recent subjets in Applied Mathematics. A chaper on american option has been added at the occasion of the IMH-IMAMIS-CIMPA school on Mathematical Finance, in Hanoi, in Spring 2007.

Here we have chosen, just as we did in our home university in Nice and Sophia-Antipolis, to introduce the main ideas of a scientific break-though that occurd during the 1970' on the New York Stock Exchange (NYSE) and the Chicago Board on Trades (CBOT), when Black and Scholes introduced their famous formula giving the way of pricing and hedging options on stocks. As it was acknoledged in 1997 by the Nobel jury, at the same time as they published the basic ideas behind it, Merton derived it using the tools of stochastic calculus, a wonderfull mathematical object to which very little mathematician where trained at this time.

Things have changed a lot in the interval ! (Almost) all young mathematician wants to know about maths applied to finance, and this is not always easy to do, at least when there is a necessity of full rigor, as stochastic calculus requires usually some confidence with Borel-Lebesgue-Kolmogorov theory of σ-algebras and filtrations by such beasts ! Of course, one may wish just to use a very brilliant spin-off of it : the famous Itô Calculus[1] and that could be tought independently from the construction of the Itô integrale. In the approach that we adopted here, we made use of an other Nobel-Prize winner : Sharp, who encouraged Cox, Ross, and Rubinstein to think of a model of stock prices based on binomial trees. So we used this clever finite model to introduce first the elementary ideas that allow, in this model, to hedge the risks, using basic linear algebra and easy-to-understand reasonning. Then, we gradually introduced the major probabilistic (or "stochastic") concepts, but again in a finite case, so to avoid the technicalities of measure theory. Even avoiding that, this is by no means trivial. Here we have collected the less immediat ideas in chapter 0, but we adopted to introduce the definitions and results gradually, when they could be used to rephrase tricks that could be understood on a financial problem. At the first glance, the stochastic version may seem a bit pedantic in the context where it is introduced (and it is !) but short after we could show how, using the formalism, one can attack problems much less obvious ! In this spirit, these lectures might be considered as *An Introduction to Mathematical Finance with Stochastic Calculus in View*, as we consider that a mathematician should both have the (geometric, information-managment, risk-hedging oriented) intuition of stochastic calculus and some ideas on the tools usually used in a rigorous introduction of the Itô integrale, that could be introduced in later work-group seminars. Actually, this way of mixing chapter 0 to the next ones is not reflected in these notes, and this will have to be done in a later version.

Again, in an attempt to help students to grasp as concretely as possible the new ideas, we used an other trick : to program small examples of financial products, using `Maple`, a

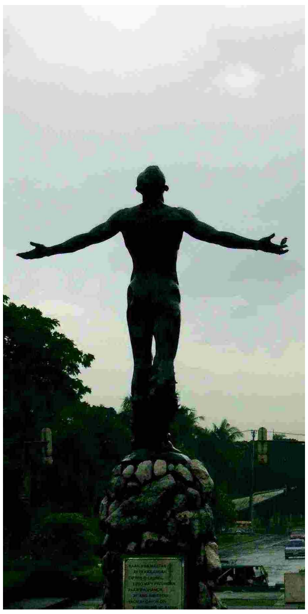
[1]Here we should also mention the tragic history of Döblin who submitted a paper to the French Academy of Sciences as Germany was to invade France, requiring the paper would not be opened for at least fifty years, in order that his ideas would not fall into the Enemy's hands, and then committed suicide.

mathematician-friendly langage with nice graphical output. Usually, the method adopted was to give a first version of the program, that would introduce the commands needed, and then to ask to adapt it in order to compute other related objects. This shows the extraordinary strength of the Cox-Ross-Rubinstein model, and explains why this model is so popular among practioners (and give accurate results, even if more effecient methods could be used, using the detour via continuous-time models). Some `Maple` exercices have been introduced in an appendix, but more are available and will be added in a later version.

In the present version, we have introduced some tools for the difficult financial problem of *Fixed Income*, as soon as one wants to take into account the risks related to this questions. Here we re-used an idea that we had tested with our Imafa[2] students in Sophia-Antipolis. This idea is the Ho and Lee model for interest-rates. It is again a binomial model, where the general idea of arbitrage-pricing in uncomplete markets introduced before can be used to price interest-rates options such as caps and floors. Here again, the ideas were tested with `Maple` programs.

These notes are to become lectures notes available on the internet on the IMAMIS web site, with the help of all colleagues in UP-Diliman that will take over the IMAMIS program.

Manila, July 12, 2005
Nice, April 19th, 2007
Marc and Francine Diener



[2]job-oriented masters program in force at the University of Nice that helped as an example in conceiving IMAMIS

Chapter 0

Conditional expectation with respect to an algebra

Given an event $A \subseteq \Omega$ such that $\mathbb{P}(A) \neq 0$ the conditional expectation $\mathbb{E}(X \mid A) := \frac{1}{\mathbb{P}(A)}\mathbb{E}(X\mathbb{I}_A)$ of the random variable (r.v.) X is a number, the (weighted) average of X on A. When letting A range over the atoms of an algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ (see below) this allows to define a new *random variable* $\overline{X}^{\mathcal{A}} = \mathbb{E}(X \mid \mathcal{A})$, the value $\overline{X}^{\mathcal{A}}(\omega)$ of which gives the best approximation of $X(\omega)$ when ω is not known, but one only knows to which $A \in \mathcal{A}$ the "state of the world" ω belongs. For the sake of simplicity we assume here that Ω is finite, and take $\mathcal{P}(\Omega)$ as the set of "events", in the sens of probability theory.

The concept of algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ allows to modelize "partial information" (in the sens that full information would be to know what $\omega^* \in \Omega$ is in force, turning the random variable X into a well specified number, namely $X(\omega^*)$) so that an $\mathcal{A}$-measurable (see below) r.v. Y is known exactly even when having only access to partial information : to which $A^* \in \mathcal{A}$ does ω^* belong.

If Ω would not be finite but infinite we would need to consider the more restrictive (and technical) case of so-called σ-algebras which would lead to more sophisticated proofs, involving measure-theoretic arguments. Nevertheless, we will only introduce objects and rules that, have conterparts in the infinite Ω case, so that the reader will get trained to a nice formal calculus, in a context where the founding ideas do not get hidden by technicalities[1].

0.1 Algebra

Definition: Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$; $\mathcal{A}$ is called an *algebra* on Ω if and only if

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$
3. For any finite set N, $(A_n)_{n\in N} \subseteq \mathcal{A}$ implies $\bigcap_{n\in N} A_n \in \mathcal{A}$.

The sets $A \in \mathcal{A}$ are called *events of the algebra* $\mathcal{A}$.

Remarks:

- When replacing finite N and Ω by infinite ones, *and N countable*, one gets the definition of a σ-algebra (see the teaching of measure theory). If Ω is finite, any algebra on Ω is also a σ-algebra.
- $\mathcal{P}(\Omega)$ and $\{\emptyset, \Omega\}$ are two elementary (and extreme) examples of algebras on Ω.

[1] On the other hand finite Ω does not allow to use computation-friendly objects such as gaussian r.v., thus loosing a lot of nice explicit computations.

- Any intersection of algebras on Ω is an algebra.
- given any family $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ of subsets of Ω one defines $< \mathcal{E} >$ to be the smallest algebra containing all sets $E \in \mathcal{E}$, algebras being ordered by inclusion. The algebra $< \mathcal{E} >$ is called the *algebra generated by* $\mathcal{E}$. Similarly, any intersection of σ-algebras on Ω is a σ-algebra ; given any familly $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ of subsets of Ω one defines $\sigma(\mathcal{E})$ to be the smallest σ-algebra containing all sets $E \in \mathcal{E}$. The algebra $\sigma(\mathcal{E})$ is called the *σ-algebra generated by* $\mathcal{E}$. If Ω is finite, $\sigma(\mathcal{E}) =< \mathcal{E} >$.
- The set $\mathcal{B}$ of all Borel sets of $\mathbb{R}^d$ is an algebra (and a σ-algebra) on $\mathbb{R}^d$.
- Let $X : \Omega \longrightarrow \mathbb{R}^d$ be a random variable. The set $\sigma(X) = \mathcal{A}_X := \{X^{-1}(B), B \in \mathcal{B}\}$ is an algebra (and a σ-algebra) called the algebra of X. As we assume that Ω is finite, one can replace $\mathcal{B}$ by $\mathcal{P}(\mathbb{R}^d)$ in the above definition (see section 0.2 below).
- As Ω is finite, any algebra $\mathcal{A}$ is generated by its *atoms* that we define using the following proposition :

Proposition 0.1 *Let $\mathcal{A}$ be an algebra on Ω ; the relation*

$$\omega' \overset{\mathcal{A}}{\sim} \omega'' \quad \textit{if and only if} \quad \forall A \in \mathcal{A},\ \omega' \in A \ \Leftrightarrow \ \omega'' \in A$$

is an equivalence relation on Ω.

Definition: The equivalence class for the equivalence relation $\overset{\mathcal{A}}{\sim}$, equivalently denoted by $\overline{\omega} = \overline{\omega}^{\mathcal{A}} = \mathrm{Atom}_{\mathcal{A}}(\omega)$, of any $\omega \in \Omega$ is called an *atom* of $\mathcal{A}$. We shall denote by $\underline{\Omega}_{\mathcal{A}}$ the partition of Ω in atoms of $\mathcal{A}$.

Remark: For any $\omega \in \Omega$, we have $\omega \in \overline{\omega}^{\mathcal{A}}$, so any atom is non-empty.

Proposition 0.2 *Assume Ω finite. If α is an atom of $\mathcal{A}$, then $\alpha \in \mathcal{A}$.*

Proof: Let $\omega \in \alpha$; observe that $\alpha = \bigcap_{\omega \in A \in \mathcal{A}} A$, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is finite as Ω is finite. $\square$

Remark: The larger the algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, the smaller are its atoms. When considering stochastic processes (family of random variables $(X_t)_{t \geq 0}$, see next chapters) we will consider "filtrations" (increasing famillies of algebras $(\mathcal{F}_t)_{t \geq 0}$). So the algebras will become larger and larger ($\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$) and thus the atoms will become smaller and smaller : this will express that when the time t increasses we get more and more information so that the atoms of "the state of the world in force ω^*" gets smaller and smaller, reducing the randomness of the future. When A_s ranges over $\underline{\Omega}_{\mathcal{F}_s}$, this gives rise to a random variable on Ω, the conditional expectation $\mathbb{E}(X_t \mid \mathcal{F}_s)$, with values $\mathbb{E}(X_t \mid A_s)$ ($s \leq t$, A_s any atom of $\mathcal{F}_s$) equal to the (weighted) average of X_t when knowing A_s (see section 0.3).

Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be an algebra on Ω and $\mathbb{P}$ a *probability on* $(\Omega, \mathcal{A})$, that is a function $\mathbb{P} : \mathcal{A} \longrightarrow [0, 1]$ such that $\mathbb{P}$ has the usual property of a probability (on $\mathcal{P}(\Omega)$) : $\mathbb{P}(\Omega) = 1$, and for any $A \in \mathcal{A}$ and $B \in \mathcal{A}$, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, and if $A \cap B = \emptyset$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Definition: Two subalgebras $\mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{G} \subseteq \mathcal{A}$ are called *independant* if and only if for any $F \in \mathcal{F}$ and any $G \in \mathcal{G}$ the r.v. $\mathbb{I}_F$ and $\mathbb{I}_G$ are independent.

Exercise: Recall that by definition the r.v. X and Y are called *independent* if and only if for any x and y $\mathbb{P}(\{X \leq x, Y \leq y\}) = \mathbb{P}(\{X \leq x\})\mathbb{P}(\{Y \leq y\})$. Show that the r.v. $\mathbb{I}_F$ and $\mathbb{I}_G$ are independent if and only if $\mathbb{P}(F \cap G) = \mathbb{P}(F)\mathbb{P}(G)$.

0.2 $\mathcal{A}$-measurable random variable

By definition, a r.v. $X : \Omega \longrightarrow \mathbb{R}^d$ is $\mathcal{A}$-measurable if and only if $X^{-1}(B) \in \mathcal{A}$ for any Borel set $B \in \mathcal{B}$. This definition is pretty technical. As we assume Ω finite, $\mathcal{A}$-measurability can fortunately be expressed more simply :

Proposition 0.3 *Let Ω be finite and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be an algebra. A r.v. $X : \Omega \longrightarrow \mathbb{R}^d$ is $\mathcal{A}$-measurable if and only if any one of the three following conditions holds :*

1. *$X^{-1}(\{b\}) \in \mathcal{A}$ for any $b \in \mathbb{R}^d$.*
2. *X is constant on any atom of $\mathcal{A}$.*
3. *it exists $a_1, \ldots, a_k$ in $\mathbb{R}$ and $A_1, \ldots, A_k$ in $\mathcal{A}$, $A_i \cap A_j = \emptyset$ if $i \neq j$, such that $X = \sum_{i=1}^k a_i \mathbb{I}_{A_i}$.*

Proof: Let's show that ($\mathcal{A}$-measurable) $\Rightarrow$ (1.) $\Rightarrow$ (2.) $\Rightarrow$ (3) $\Rightarrow$ ($\mathcal{A}$-measurable), which obviously shows that all implications are equivalences.

[**($\mathcal{A}$-measurable) $\Rightarrow$ (1.)**] : $\mathcal{B}$ is an algebra containing all closed subsets $B \subseteq \mathbb{R}^d$; as $\{b\}$ is closed, $X^{-1}(\{b\}) \in \mathcal{A}$ as X is $\mathcal{A}$-measurable.

[**(1.) $\Rightarrow$ (2.)**] : Let $A := \overline{\omega}^{\mathcal{A}}$ be the atom of any $\omega \in \Omega$, $b := X(\omega)$, and $\omega' \in A$; by assumption $\omega \in X^{-1}(\{b\}) \in \mathcal{A}$ so $\omega \in A \subseteq X^{-1}(\{b\}) \in \mathcal{A}$, as $A = \overline{\omega}^{\mathcal{A}}$. Now, as $\omega' \overset{\mathcal{A}}{\sim} \omega$, $\omega' \in A \subseteq X^{-1}(\{b\})$, thus $X(\omega') = b = X(\omega)$.

[**(2.) $\Rightarrow$ (3.)**] : As Ω is finite, so is $X(\Omega)$; let $X(\Omega) = \{a_1, \ldots, a_k\}$ be these values and $A_i = X^{-1}(a_i)$. As X is constant on the atoms of $\mathcal{A}$, each A_i is a (necessarily finite) union of atoms and is thus in $\mathcal{A}$. We see that for any $i = 1 \ldots k$ and any $\omega \in A_i$ we have

$$\left(\sum_{i=1}^k a_i \mathbb{I}_{A_i}\right)(\omega) = \sum_{i=1}^k a_i \mathbb{I}_{A_i}(\omega) = a_i.$$

Thus $\sum_{i=1}^k a_i \mathbb{I}_{A_i} = X$.

[**(3.) $\Rightarrow$ ($\mathcal{A}$-measurable)**] : Let $B \in \mathcal{B}$ be any any Borel set ; we have

$$X^{-1}(B) = X^{-1}(B \cap X(\Omega)) = X^{-1}(\{b_1\}) \cup X^{-1}(\{b_2\}) \cup \ldots \cup X^{-1}(\{b_n\})$$

where $\{b_1, b_2, \ldots, b_n\} = B \cap X(\Omega)$ which is finite as Ω is finite. As by assumption $X = \sum_{i=1}^k a_i \mathbb{I}_{A_i}$, any $X^{-1}(\{b_j\}) = \bigcup_{a_i = b_j} A_i$ is a union of (finitely many) atoms, which shows that $X^{-1}(B)$ is a finite union of atoms and thus belongs to $\mathcal{A}$ as $\mathcal{A}$ is an algebra.

□

Remarks:

- It are of course the two last parts of the proof that use the finiteness of Ω. Observe that actually we only need that $X(\Omega)$ is finite : this may be usefull when using infinite sequences of finitely valued r.v..
- Let $X : \Omega \longrightarrow \mathbb{R}^d$ be any r.v. on some finite Ω. The atoms of $\sigma(X)$ are the $X^{-1}\{b\}$, for all $b \in X(\Omega)$.
- Any r.v. X is $\sigma(X)$-measurable. The r.v. X and Y are independent if and only if their algebras $\sigma(X)$ and $\sigma(Y)$ are independent.
- It is a common and usefull abuse to write $X \in \mathcal{A}$ for "the r.v. X is $\mathcal{A}$-measurable". So, for instance, $X \in \sigma(X)$.

- If $X \in \mathcal{A}$ and $Y \in \mathcal{A}$, then $aX + bY \in \mathcal{A}$ and $XY \in \mathcal{A}$ for any real numbers a and b.

Theorem 0.4 *Let $X : \Omega \longrightarrow \mathbb{R}^d$ be any r.v. and $\sigma(X)$ be its algebra. Any r.v. $Y : \Omega \longrightarrow \mathbb{R}^{d'}$ is $\sigma(X)$-measurable if and only if it exists $f : \mathbb{R}^d \longrightarrow \mathbb{R}^{d'}$ such that $Y = f(X)$.*

Proof: As X is $\sigma(X)$-measurable, X is constant on the atoms of $\sigma(X)$ and thus so is $Y = f(X)$. Conversly, if Y is $\sigma(X)$-measurable, then it is constant on the atoms $X^{-1}(\{b\})$ ($b \in X(\Omega)$) of $\sigma(X)$; for each $b \in X(\Omega)$, let c be this constant value and define $f(b) := c$; this defines f on $X(\Omega)$; finally, for $x \in X(\Omega)^c$ let $f(x) = 0$. Clearly $Y = f(X)$ by construction. $\square$

0.3 Properties of the conditional expectation

Recall that we assume that Ω is finite. Thus, for any r.v. $X : \Omega \longrightarrow \mathbb{R}$,

$$\mathbb{E}(X) = \sum_{\omega\in\Omega} \mathbb{P}(\{\omega\})X(\omega) = \sum_{\omega\in\Omega'} \mathbb{P}(\{\omega\})X(\omega),$$

where $\Omega' \subseteq \Omega$ is the subset of all ω such that $\mathbb{P}(\{\omega\}) \neq 0$. So, without loss of generality, *we* can and will *assume that* $\mathbb{P}(\{\omega\}) \neq 0$ for any $\omega \in \Omega$, so, for any algebra $\mathcal{A}$, the atom $\overline{\omega}^{\mathcal{A}}$ of any $\omega \in \Omega$ has non-zero probability. This makes it easy to define the conditional expectation with respect to $\mathcal{A}$ in the following way :

Definition: Let X be a r.v. on Ω and $\mathcal{A}$ be an algebra on Ω. The *conditional expectation of* X with respect to $\mathcal{A}$ is the r.v. $\overline{X}^{\mathcal{A}}$ denoted by $\mathbb{E}(X \mid \mathcal{A})$ defined, for all $\omega \in \Omega$, by

$$\overline{X}^{\mathcal{A}}(\omega) = \mathbb{E}(X \mid \mathcal{A})(\omega) := \mathbb{E}(X \mid \overline{\omega}^{\mathcal{A}}) = \frac{1}{\mathbb{P}(\overline{\omega}^{\mathcal{A}})}\mathbb{E}(X\mathbb{I}_{\overline{\omega}^{\mathcal{A}}})$$

Remark: It has been possible to give the above definition of the conditional expectation with respect to an algebra thanks to the hypothesis that Ω is finite ; an other way to define the conditional expectation with respect to an algebra is just

$$\mathbb{E}(X \mid \mathcal{A}) = \sum_{A_i\in\Omega_{\mathcal{A}}} a_i\mathbb{I}_{A_i} \text{ with } a_i := \mathbb{E}(X \mid A_i) = \frac{\mathbb{E}(X\mathbb{I}_{A_i})}{\mathbb{P}(A_i} = \frac{\mathbb{E}(X\mathbb{I}_{A_i})}{\mathbb{E}(\mathbb{I}_{A_i})}. \tag{1}$$

Proposition 0.6 below gives a list of properties of the conditional expectation that are still valid in a much more general case and that suffice generaly to proove the results used in mathematical finance, without coming back to the definition.

Let us first show a fondamental lemma on conditional expectation :

Lemma 0.5 *The conditional expectation of X with respect of $\mathcal{A}$ $\overline{X}^{\mathcal{A}}(\omega) = \mathbb{E}(X \mid \mathcal{A})(\omega)$ is the unique $\mathcal{A}$-measurable r.v. $\overline{X}$ such that, for any event $E \in \mathcal{A}$,*

$$\mathbb{E}(\overline{X}\mathbb{I}_E) = \mathbb{E}(X\mathbb{I}_E) \tag{2}$$

Proof: Uniqueness : Assume $\overline{X}$ is any $\mathcal{A}$-measurable r.v. such that (2) holds for any $E \in \mathcal{A}$; as $\overline{X}$ is $\mathcal{A}$-measurable, it has to be constant on the atoms of $\mathcal{A}$. Let A be any atom of $\mathcal{A}$; let $a := \overline{X}(\omega)$, $\omega \in A = \overline{\omega}^{\mathcal{A}}$ be this value. As $X\mathbb{I}_A = a\mathbb{I}_A$, applying (2) one gets

$$a\mathbb{P}(A) = a\mathbb{E}(\mathbb{I}_A) = \mathbb{E}(a\mathbb{I}_A) = \mathbb{E}(\overline{X}\mathbb{I}_A) = \mathbb{E}(X\mathbb{I}_A)\ ,$$

$$\text{thus, as } A = \overline{\omega}^{\mathcal{A}},\ \overline{X}(\omega) = a = \frac{1}{\mathbb{P}(A)}\mathbb{E}(X\mathbb{I}_A) = \frac{1}{\mathbb{P}(\overline{\omega}^{\mathcal{A}})}\mathbb{E}(X\mathbb{I}_{\overline{\omega}^{\mathcal{A}}}) = \mathbb{E}(X \mid \mathcal{A})(\omega),$$

thus $\overline{X} = \overline{X}^{\mathcal{A}}$, which shows uniqueness.

Now, by definition, $\mathbb{E}(X \mid \mathcal{A})$ is constant on the atoms of $\mathcal{A}$, thus is $\mathcal{A}$-measurable. Finally, let E be any event belonging to $\mathcal{A}$ and let $E = A_1 \dot\cup \ldots \dot\cup A_n$ be its decomposition in atoms of $\mathcal{A}$. We have

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(X \mid \mathcal{A})\mathbb{I}_E) &= \mathbb{E}(\overline{X}^{\mathcal{A}}\mathbb{I}_E) = \mathbb{E}(\overline{X}^{\mathcal{A}}(\mathbb{I}_{A_1} + \ldots + \mathbb{I}_{A_n})) \\
&= \sum_{i=1}^{n} \mathbb{E}(\overline{X}^{\mathcal{A}}\mathbb{I}_{A_i}) = \sum_{i=1}^{n} \mathbb{E}(a_i\mathbb{I}_{A_i})\ ,\ \text{with } a_i := \frac{1}{\mathbb{P}(A_i)}\mathbb{E}(X\mathbb{I}_{A_i}) \\
&= \sum_{i=1}^{n} a_i\mathbb{E}(\mathbb{I}_{A_i}) = \sum_{i=1}^{n} \frac{1}{\mathbb{P}(A_i)}\mathbb{E}(X\mathbb{I}_{A_i})\mathbb{P}(A_i) \\
&= \sum_{i=1}^{n} \mathbb{E}(X\mathbb{I}_{A_i}) = \mathbb{E}\left(X\sum_{i=1}^{n}\mathbb{I}_{A_i}\right) = \mathbb{E}(X\mathbb{I}_A).
\end{aligned}$$

which shows that $\mathbb{E}(X \mid \mathcal{A})$ satisfies 2 for any $E \in \mathcal{A}$. □

Proposition 0.6 *Let a and b be numbers, X and Y be r.v. on Ω, and $\mathcal{F} \subseteq \mathcal{G}$ be algebras on Ω. Following relations hold :*

1. $\mathbb{E}(X \mid \mathcal{P}(\Omega)) = X$ *and* $\mathbb{E}(X \mid \{\emptyset, \Omega\}) = \mathbb{E}X$;
2. $\mathbb{E}(aX + bY \mid \mathcal{F}) = a\mathbb{E}(X \mid \mathcal{F}) + b\mathbb{E}(Y \mid \mathcal{F})$ *and* $\mathbb{E}(\mathbb{I}_\Omega \mid \mathcal{F}) = 1$;
3. *If* $X \geq 0$ *then* $\mathbb{E}(X \mid \mathcal{F}) \geq 0$, *and as* $\mathbb{P}(\{\omega\}) \neq 0$ *for any* $\omega \in \Omega$, $X > 0 \Rightarrow \mathbb{E}(X \mid \mathcal{F}) > 0$;
4. $Y \in \mathcal{F} \Rightarrow \mathbb{E}(XY \mid \mathcal{F}) = Y\mathbb{E}(X \mid \mathcal{F})$;
5. $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathbb{E}(\mathbb{E}(X \mid \mathcal{G})\ \mid\ \mathcal{F}) = \mathbb{E}(X \mid \mathcal{F})$;
6. *If $\mathcal{F}$ is independent of $\sigma(X)$ then* $\mathbb{E}(X \mid \mathcal{F}) = \mathbb{E}X$.

Proof: The principle of the proof of all these properties consists in showing that the proposed value of the conditional expectation with respect to the algebra $\mathcal{F}$ under consideration is an $\mathcal{F}$-measurable r.v. satisfying the caracteristic property (2). As this caracteristic property is also true in the case of infinite Ω, the proofs given below will also show proposition 0.6 in teh general case.

1. Let $\overline{X} := X$; as $\mathcal{P}(\Omega)$ is the largest possible algebra on Ω, $\overline{X}$ is obviously $\mathcal{P}(\Omega)$-measurable. Let $E \in \mathcal{P}(\Omega)$; now $\mathbb{E}(\overline{X}\mathbb{I}_E) = \mathbb{E}(X\mathbb{I}_E)$ by definition of $\overline{X}$, and as this is true for any $E \in \mathcal{P}(\Omega)$, this shows that $X =: \overline{X} = \mathbb{E}(X \mid \mathcal{P}(\Omega))$.

   In order to show that $\mathbb{E}(X \mid \{\emptyset, \Omega\}) = \mathbb{E}X$ let $\overline{X} := \mathbb{E}X$; for $E := \emptyset$, $\mathbb{E}(\overline{X}\mathbb{I}_E) = \mathbb{E}(\mathbb{E}(X) \cdot 0) = 0 = \mathbb{E}(X\mathbb{I}_E)$; for $E := \Omega$ we have $\mathbb{E}(\overline{X}\mathbb{I}_E) = \mathbb{E}(\mathbb{E}X \cdot 1) = \mathbb{E}X = \mathbb{E}(X\mathbb{I}_E)$. This shows that $\mathbb{E}X =: \overline{X} = \mathbb{E}(X \mid \{\emptyset, \Omega\})$.

2. Let $\overline{X} := a\mathbb{E}(X \mid \mathcal{F}) + b\mathbb{E}(Y \mid \mathcal{F})$, which is $\mathcal{F}$-measurable as $\mathbb{E}(X \mid \mathcal{F})$ and $\mathbb{E}(Y \mid \mathcal{F})$ are so. Now, let $E \in \mathcal{F}$; we have

$$\begin{aligned}
\mathbb{E}(\overline{X}\mathbb{I}_E) &= \mathbb{E}((a\mathbb{E}(X \mid \mathcal{F}) + b\mathbb{E}(Y \mid \mathcal{F}))\mathbb{I}_E) \\
&= a\mathbb{E}(\mathbb{E}(X \mid \mathcal{F})\mathbb{I}_E) + b\mathbb{E}(\mathbb{E}(Y \mid \mathcal{F})\mathbb{I}_E) \\
&= a\mathbb{E}(X\mathbb{I}_E) + b\mathbb{E}(Y\mathbb{I}_E) \text{ by } (2) \text{ , as } E \in \mathcal{F} \\
&= \mathbb{E}((aX + bY)\mathbb{I}_E).
\end{aligned}$$

As this is true for any $E \in \mathcal{F}$, this shows that $X =: a\mathbb{E}(X \mid \mathcal{F}) + b\mathbb{E}(Y \mid \mathcal{F}) = \mathbb{E}(aX + bY \mid \mathcal{F})$.

3. Let $\omega \in \Omega$; assume $X \geq 0$. Then $\mathbb{E}(X \mid \mathcal{F})(\omega) = \frac{1}{\mathbb{P}(\overline{\omega}^{\mathcal{F}})}\mathbb{E}(X\mathbb{I}_{\overline{\omega}^{\mathcal{F}}}) \geq 0$ and is non-zero if $X > 0$. As this is true for any $\omega \in \Omega$, this shows that $\mathbb{E}(X \mid \mathcal{F}) \geq 0$ and $\mathbb{E}(X \mid \mathcal{F}) > 0$ if $X > 0$.

4. Let $\overline{Z} := Y\mathbb{E}(X \mid \mathcal{F})$; as Y and $\mathbb{E}(X \mid \mathcal{F})$ are $\mathcal{F}$-measurable, so is $\overline{Z}$. Let $E \in \mathcal{F}$ and A_1, $\ldots$, A_n be the atoms of $\mathcal{F}$ contained in E ; thus E is the disjoint union of $A_1, \ldots, A_n$ and $\mathbb{I}_E = \sum_{i=1}^n \mathbb{I}_{A_i}$. As Y is $\mathcal{F}$-measurable, it is constant on the atoms of $\mathcal{F}$, so it exists constants $y_1, \ldots, y_n$ such that $Y(\omega) = y_i$ for any $\omega \in A_i$. Now

$$\begin{aligned}
\mathbb{E}(\overline{Z}\mathbb{I}_E) &= \mathbb{E}\left(Y\mathbb{E}(X \mid \mathcal{F})\sum_{i=1}^n \mathbb{I}_{A_i}\right) = \sum_{i=1}^n \mathbb{E}(Y\mathbb{E}(X \mid \mathcal{F})\mathbb{I}_{A_i}) \\
&= \sum_{i=1}^n \mathbb{E}(y_i\mathbb{E}(X \mid \mathcal{F})\mathbb{I}_{A_i}) = \sum_{i=1}^n y_i\mathbb{E}(\mathbb{E}(X \mid \mathcal{F})\mathbb{I}_{A_i}) \\
&= \sum_{i=1}^n y_i\mathbb{E}(X\mathbb{I}_{A_i}) \text{ , as } A_i \in \mathcal{F}, \\
&= \mathbb{E}\left(\sum_{i=1}^n y_i X\mathbb{I}_{A_i}\right) = \mathbb{E}\left(\sum_{i=1}^n XY\mathbb{I}_{A_i}\right) \\
&= \mathbb{E}\left(XY\sum_{i=1}^n \mathbb{I}_{A_i}\right) = \mathbb{E}(XY\mathbb{I}_E).
\end{aligned}$$

As this is true for any $E \in \mathcal{F}$, this shows that $Y\mathbb{E}(X \mid \mathcal{F}) =: \overline{Z} = \mathbb{E}(XY \mid \mathcal{F})$.

5. Let $\overline{X} := \mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{F})$ that is $\mathcal{F}$-measurable as being a conditional expection with respect to $\mathcal{F}$. Let $E \in \mathcal{F}$; we have

$$\begin{aligned}
\mathbb{E}(\overline{X}\mathbb{I}_E) &= \mathbb{E}(\mathbb{E}((\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{F})\mathbb{I}_E) \\
&= \mathbb{E}(\mathbb{E}(X \mid \mathcal{G})\mathbb{I}_E) \text{ , as } E \in \mathcal{F} \\
&= \mathbb{E}(X\mathbb{I}_E) \text{ , as } E \in \mathcal{F} \subseteq \mathcal{G} \text{ thus } E \in \mathcal{G}.
\end{aligned}$$

As this is true for any $E \in \mathcal{F}$, this shows that $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{F}) =: \overline{X} = \mathbb{E}(X \mid \mathcal{F})$.

6. Assume that $\mathcal{F}$ is independent of $\sigma(X)$. Let $\overline{X} := \mathbb{E}X$ and let $E \in \mathcal{F}$. As $\mathcal{F}$ is independent of $\sigma(X)$, the r.v. $\mathbb{I}_E$ is independant of X. Now

$$\begin{aligned}
\mathbb{E}(\overline{X}\mathbb{I}_E) &= \mathbb{E}(\mathbb{E}(X)\mathbb{I}_E) = \mathbb{E}(X)\mathbb{E}(\mathbb{I}_E) \\
&= \mathbb{E}(X\mathbb{I}_E) \text{ , as } X \text{ is independent of } \mathbb{I}_E.
\end{aligned}$$

As this is true for any $E \in \mathcal{F}$, this shows that $\mathbb{E}(X) =: \overline{X} = \mathbb{E}(X \mid \mathcal{F})$.

□

Exercise 0.1 *Give a proof for the various remarks and unshown proposition above.*

Exercise 0.2 *Define* $\mathbb{E}(Y \mid X) := \mathbb{E}(Y \mid \sigma(X))$. *Show that* $\mathbb{E}(Y \mid \sigma(X)) = f(X)$, *for*

$$f(x) := \mathbb{E}(Y \mid \{X = x\}) := \frac{1}{\mathbb{P}\{X = x\}} \mathbb{E}(Y \mathbb{I}_{\{X=x\}}).$$

Exercise 0.3 *Assume* X *and* Y *are independent r.v.. Show that* $\mathbb{E}(X \mid Y) = \mathbb{E}X$, *and* $\mathbb{E}(g(X, Y) \mid X) = f(X)$, *for* $f(x) := \mathbb{E}(g(x, Y))$.

Exercise 0.4 Jensen's Inequality : *Assume* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *is convex. Show that*

$$\varphi(\mathbb{E}(X \mid \mathcal{A})) \leq \mathbb{E}(\varphi(X) \mid \mathcal{A}) \text{ for any algebra } \mathcal{A}.$$

Hint : *as* φ *is convex it exists a sequence* $(a_n, b_n)_{n \in \mathbb{N}}$ *such that* $\varphi(x) = Sup_{n \in \mathbb{N}}(a_n x + b_n)$ *; use Proposition 0.6-3.*

Chapter 1

Vanilla options

1.1 Basic approach

1.1.1 What are derivatives ?

In some sens we can say that the modern theory of mathematical finance (the Black-Scholes theory) begins with giving a solution to the problem of pricing some financial risks and being able to hedge them. What kind of financial risks are under consideration here ?

Let us take an example : suppose that the owner of a firm knows that he will need to buy some commodity involved in his production after, say, two months (for example wood if the firm produces furniture, paper if it produces books, cacao if it produces chocolate, . . .), but the price of this commodity varies a lot, in an unpredictible way. He is not a speculator, he would prefere not to face such a risk. As all of us when we subscribe an insurance contracts, he is ready to pay now a small amont in order to have the right to buy the commodity he needs at the precise date he knows he will need it and at some fixed price, the *exercice price* (for example today's price). Such a contract is called an option, a *call option* if it is about the right to buy and a *put option* if it is about the right to sell.

Call and put options are the simplest examples of more general financial contracts called *derivatives*. This name is related to the fact that their values, at the exercise date, depends on the value of some *underlying asset* (the commodity in our example) at that date. The most commun options are written on financial assets, like stocks or bonds. The value of an option at the exercise date is called its *payoff*. Figure 1.1.1 shows the payoff of a call and the one of a put, as functions of the underlying asset. For exemple for a call, it is worth zero if the value S of the underlying asset is less or equal to the exercise price K because the owner of the option will not exercise his right to buy S at price K if he can buy it for less, and is worth the difference $S - K$ if S is greater than K because when added to the exercise price K one gets the actual price S.

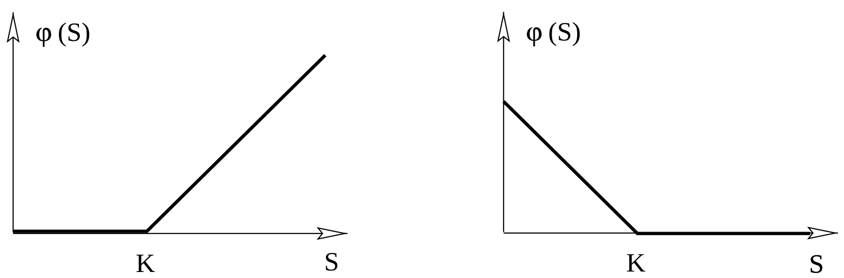


Figure 1.1: Payoff of a call $\varphi(S) = (S - K)^+$ and payoff of a put $\varphi(S) = (K - S)^+$.

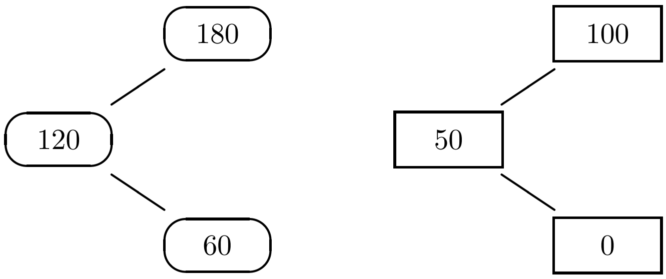


Figure 1.2: A one step model of an underlying asset (left) and the corresponding model of a call option with exercise price $K = 80$ (assuming $R = 1$) (right).

1.1.2 Option pricing

Option contracts exist since a long time ; for example, more than hundred years ago, farmers subscribed put options buying the right to sell, during the following summer, their crops at some fixed (minimal) price. The true novelty of the Black Scholes theory was to provide a mathematical tool to compute a fair price for such a contract. As a consequence, options who were only *over the counter* contracts (contracts between a producer and a financier), have been introduced after the Black-Scholes discovery as financial securities on markets. They are now commonly exchanged in most of the financial places in the world. Think of an insurance contract that once it has been underwritten can be sold back at any time for possibly an increased price.

In order to compute the price of an option, one needs first to choose a model for the dynamic of the underlying asset. The Cox-Ross-Rubinstein model (CRR) that we will present now, was introduce shortly after the original Black-Scholes model (BS), as a simplified version of it. It is easy to understand because the mathematics required are at high school level and nevertheless almost all important concepts of the Black-Scholes theory are already present in this simplified model. We will begin with the one step model, then the two steps model who are toy models and finally the general (n steps) CRR model.

The one step model : hedging portfolio

Let $t = 0$ be today and $t = T$ be the exercise date of an option. In the one step model, we do not consider intermediate dates between 0 and T. Let S_0 be today's price of the underlying asset. We know neither the future price S_T, nor the option's payoff as it is a function of S_T that we shall denote by $\varphi(S_T)$. As a first (unrealistic) model, we assume that S_T can only take two values $S_T = S_0u$ or $S_T = S_0d$ (u for up and d for down) and we assume that the value of an amount of 1 Euro today will be worth R Euros at time T ($1/R$ is the discount factor for time interval $[0, T]$). Suppose also $0 < d < R < u$ and $1 < R$.

If for example the option is a call option and the exercise price K is such that $S_0d \leq K \leq S_0u$, the seller (also called *writter*) of the option will have to pay either $S_0u - K$ if $S_T = S_0u$ or nothing if $S_T = S_0d$. What can she do to be able to fulfil her obligations ? The idea is simple. As she will be in bad shape in the case the underlying asset price increases, she can hedge this risk by buying at time $t = 0$ a convenient amount of the underlying security : she will grow rich when S increases. But which amount ? Let x denotes this amount. Consider a portfolio (x, y) containing a quantity x of underlying asset S_0 and a quantity y of Euros. Its time 0 value is $xS_0 + y$ and its time T value is either $xS_0u + yR$ or $xS_0d + yR$, depending on whether $S_T = S_0u$ or $S_T = S_0d$. The idea is to choose x and y such that the value of the portfolio at time T is precisely the payoff of the option, $S_0u - K$ or 0 in our example. Such a portfolio is called an

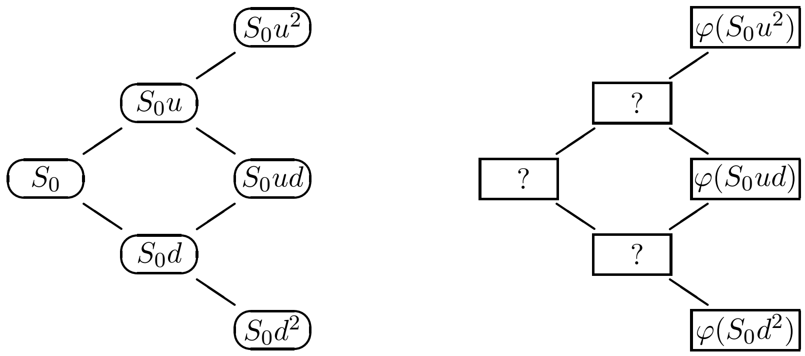


Figure 1.3: Exemple of a two steps model : underlying asset (left) and option (right)

hedging portfolio. Its componants x and y satisfy the equations

$$\left\{ \begin{array}{lcl} xS_0u + yR & = & \varphi(S_0u) \\ xS_0d + yR & = & \varphi(S_0d) \end{array} \right. \tag{1.1}$$

Obviously, this linear system has a unique solution, given by

$$\left\{ \begin{array}{lcl} x & = & \dfrac{\varphi(S_0u) - \varphi(S_0d)}{S_0(u-d)} \\ y & = & \frac{1}{R}\dfrac{u\varphi(S_0d) - d\varphi(S_0u)}{u-d} \end{array} \right. \tag{1.2}$$

The time 0 value of the portfolio $xS_0 + y$ is called the *premium* of the option : it is the price that has to be payed by the buyer to the seller for the contract.

Figure 1.2 shows an example of a call with exercise price $K = 80$: the three values 120, 180 and 60 correspond to the model of the underlying asset, the two final values 100 and 0 correspond to the payoff of the option ($\varphi(S_T) = (S_T - 80)^+$) and the initiale value $50 = 120x + y$ is easy to compute from the solutions (x, y) of

$$\left\{ \begin{array}{lcl} 180x + y & = & 100 \\ 60x + y & = & 0 \end{array} \right. \tag{1.3}$$

assuming $R = 1$ for simplicity. The solution is $(x, y) = (\frac{5}{6}, -50)$. Thus, at time 0, the seller of the call option builds up a portfolio containing $\frac{5}{6}$ units of S (that cost $\frac{5}{6}120 = 100$ Euros each) using the 50 Euros of the premium and a 50 Euros loan. At time T, the value of this portfolio will be, in each of the two cases, exactly equal to the amount he has to pay to the owner of the option.

The two steps model : dynamic hedging

On its own, the idea of a hedging portfolio (x, y) built at time $t = 0$ once for all, is not enough in a multistep model when more than two possible asset prices exist at the exercise date. But if one adds the possibility to rebalance the portfolio at the intermediate date, a solution exists : this is called *dynamic hedging* and we will detail it now.

Let us consider a model of an underlying asset with two time steps $t \in \{0, \delta t, 2\delta t = T\}$ instead of one : assume the underlying asset (S_t) equals S_0 at the initial date, can take one of the two values $S_{\delta t} = S_0d$ or $S_{\delta t} = S_0u$ at time $t = \delta t$ and one of the three values $S_T = S_0d^2$, $S_T = S_0ud$ or $S_T = S_0u^2$ at time T (see figure 1.3). To build a hedging portfolio for an option written on (S_t) with payoff $\varphi(S_T)$, we first remark that its three different values Π_T at the end

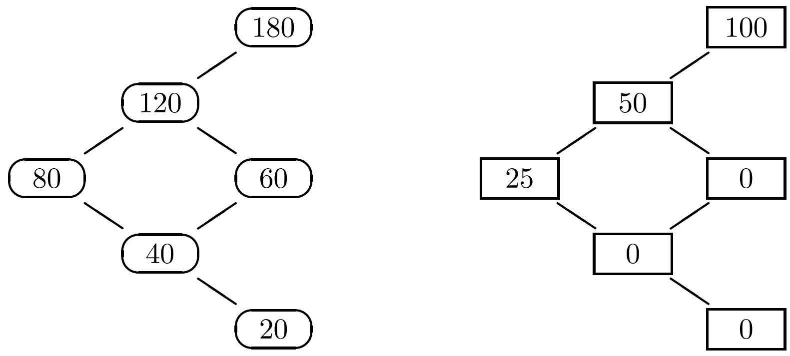


Figure 1.4: A two steps model of an underlying asset (left) and the corresponding model of a call option with exercise price $K = 80$ (assuming $R = 1$) (right).

are known, $\Pi_T = \varphi(S_T)$, and its two different values $\Pi_{\delta t} = x_{\delta t} S_{\delta t} + y_{\delta t}$ at intermediate date can be deduced from Π_T exactly as in the one step model by solving the system of equations :

$$\left\{ \begin{array}{rcl} xS_0u^2 + yR &=& \varphi(S_0u^2) \\ xS_0ud + yR &=& \varphi(S_0ud) \end{array} \right. \tag{1.4}$$

for the upper value and the following system for the lower :

$$\left\{ \begin{array}{rcl} xS_0ud + yR &=& \varphi(S_0ud) \\ xS_0d^2 + yR &=& \varphi(S_0d^2). \end{array} \right. \tag{1.5}$$

Let $\Pi^u_{\delta t}$ et $\Pi^d_{\delta t}$ denote the upper and lower values of $\Pi_{\delta t}$ that can be computed replacing $x_{\delta t}$ and $y_{\delta t}$ in $\Pi^u_{\delta t} = x_{\delta t} S_0 u + y_{\delta t}$ (resp. in $\Pi^d_{\delta t} = x_{\delta t} S_0 d + y_{\delta t}$) by the solution of (1.4) (resp. the solution of (1.5)). Once these two values have been computed, the initial portfolio value Π_0, which is also the option premium, can be obtained by solving the system :

$$\left\{ \begin{array}{rcl} xS_0u + yR &=& \Pi^u_{\delta t} \\ xS_0d + yR &=& \Pi^d_{\delta t} \end{array} \right. \tag{1.6}$$

and letting $\Pi_0 = x_0 S_0 + y_0$, where (x_0, y_0) is its solution.

Example: Suppose the evolution of (S_t) is given by

$$S_0 = 80 \quad \text{becomes} \quad S_{\delta t} = 120 \text{ or } S_{\delta t} = 40 \tag{1.7}$$

$$S_{\delta t} = 120 \quad \text{becomes} \quad S_{2\delta t} = 180 \text{ or } S_{2\delta t} = 60 \tag{1.8}$$

$$S_{\delta t} = 40 \quad \text{becomes} \quad S_{2\delta t} = 60 \text{ or } S_{2\delta t} = 20 \tag{1.9}$$

and consider a call option with exercise date $T = 2\delta t$ and exercise price $K = 80$. Assume for simplicity, as in the previous example, the interest rate factor R equal to 1. The price of the hedging portfolio is known for $t = 2\delta t = T$, $\Pi_{2\delta t} = (S_{2\delta t} - 80)^+$. Using the previous example, it is worth $\Pi^u_{\delta t} = 50$ when $S_{\delta t} = 120$. Without any new computations, it is worthless when $S_{\delta t} = 40$ because the two possible values of the portfolio for next date are 0, then $x_{\delta t} = y_{\delta t} = 0$ (there is no more risk to hedge in this case). At time $t = 0$ the hedging portfolio(x_0, y_0) has to satisfy the equation $x_0 S_{\delta t} + y_0 = \Pi_{\delta t}$, and thus the system :

$$\left\{ \begin{array}{rcl} x_0 120 + y_0 &=& x_0 S_0 u + y_0 = 50 \\ x_0 60 + y_0 &=& x_0 S_0 d + y_0 = 0. \end{array} \right. \tag{1.10}$$

The solution is $x_0 = \frac{5}{8}$ and $y_0 = -25$ and thus $\Pi_0 = \frac{5}{8} 80 - 25 = 25$. The seller of the option receives the premium $\Pi_0 = 25$ at time $t = 0$, she takes out a 25 Euros loan in order to buy $\frac{5}{8}$

units of asset for 80 each. At the end of the first period $t = \delta t$ there are two possibilities : in the down state where the asset price is $S_{\delta t} = 40$, she clears the portfolio : its first componant $x_0 S_{\delta t} = \frac{5}{8}40 = 25$ allows exactly to pay off the debt $y_0 = 25$; in the up state, $S_{\delta t} = 120$, the portfolio has to hold $x_{\delta t} = \frac{5}{6}$ units of asset, according to the computations we have done above ; as it holds already $\frac{5}{8}$ units, she must buy $\frac{5}{6} - \frac{5}{8} = \frac{10}{48}$ units more at $S_{\delta t} = 120$ each, and takes a new $\frac{10}{48}120 = 25$ Euros loan, increasing her debt up to $25+25 = 50$. Rebalancing her position in such a way, the option seller hedges exactly her risk because her portfolio will have at the end the precise value she will need in each of the three cases.

Remark: To show why dynamic hedging is useful, it is interesting to observe in this example what happens if the seller does not hedge the option, either taking the premium and doing nothing else or even building the hedging portfolio but never rebalancing it. If she just takes the premium, she will still have 25 (under our hypothesis $R = 1$) at the end and will not have the 100 Euros she has to pay to the option's owner in the case where $S_T = 180$. With a portfolio of 5/8 units of S_t and a debt of 25, built at initial date, she will not succeed to hedge her risk in every cases without rebalancing it : indeed, such a portfolio is worth at exercise date T :

- $\frac{5}{8}180 - 25 = 87,5$, if the asset price is 180 while she has to pay 100.
- $\frac{5}{8}60 - 25 = 12,5$, if the asset price is 60 while she has nothing to pay ; no problem in that case.
- $\frac{5}{8}20 - 25 = -12,5$, if the asset price is 20 ; she has nothing to pay but she is not able to pay off her debt.

The CRR model : risk neutral probability

By now, it should be clear that we can generalize the model to an arbitrary number n of time periods. This was done by J. Cox, S. Ross, and M. Rubinstein in 1979 [1]. They proposed to model the price of an asset by :

- n successive dates between 0 and T, $\mathbb{T} := [0..T]_{\delta t} = \{0, \delta t, 2\delta t, \ldots, n\delta t = T\}$, with $\delta t > 0$ given (and often assumed to be small)
- a risky asset $(S_t)_{t\in\mathbb{T}}$ with one value S_0 at time $t = 0$, and which follows a random walk defined by induction : if S_t denotes the asset price at time $t \in \mathbb{T}\backslash\{n\delta t\}$, its value at time $t + \delta t$ is either $S_t u$ or $S_t d$, where u and d are such that $0 < d < u$. In other words, $(S_t)_{t\in\mathbb{T}}$ belongs to a binary tree with $k + 1$ nodes for $t = k\delta t \in \mathbb{T}$ and $k + 1$ values $\{S_0 u^j d^{k-j}, j = 0, \ldots, k\}$, where j stands for the number of "up" between the initial date and $t = k\delta t$.
- a non risky asset defined on $\mathbb{T}$ by $B_0 = 1$ and $B_t = B_{t-\delta t} e^{r\delta t} = e^{rt}$ where r is the interest rate for a time interval of length δt. As we do in the one and two steps models, the quantity $R = e^{r\delta t}$ is assumed to be constant.

To price and hedge a call option with exercise date $T = n\delta t$ and exercise price K written on such an asset $(S_t)_{t\in\mathbb{T}}$, it is easy to generalize the case of the two steps model. The hedging portfolio $(\Pi_t)_{t\in\mathbb{T}}$ is known, by definition, at the exercise date T and can be defined for all $t < T$ by backward induction : suppose we know its different values at time $t + \delta t$, let Π be one of these values at time t corresponding to $S_t = S$ and $B_t = B$, and let Π^u and Π^d its two possible values for the next date. The two components (x, y) of Π are the solution of

$$\begin{aligned} xSu + ye^{r\delta t}B &= \Pi^u \\ xSd + ye^{r\delta t}B &= \Pi^d \end{aligned}$$

that is

$$x = \frac{\Pi^u - \Pi^d}{Su - Sd} \text{ and } y = e^{-r\delta t}\frac{\Pi^d u - \Pi^u d}{B(u-d)}. \tag{1.11}$$

and thus $\Pi = xS + yB$. The crucial remark is the following : it is possible to rewrite this portfolio value as $\Pi = xS + y = \frac{\Pi^u - \Pi^d}{Su - Sd}S + e^{-r\delta t}\frac{\Pi^d u - \Pi^u d}{B(u-d)}$, thus

$$\Pi = e^{-r\delta t}(p\Pi^u + q\Pi^d), \tag{1.12}$$

introducing the quantities

$$p := \frac{e^{r\delta t} - d}{u - d} \text{ and } q := \frac{u - e^{r\delta t}}{u - d}. \tag{1.13}$$

These quantities satisfy $p + q = 1$ and moreover $0 < p < 1$ and $0 < q < 1$ provided $0 < d < e^{r\delta t} < u$. Thus, if we consider Π^u et Π^d as the possible values of a two-valued random variable Π with $P(\Pi = \Pi^u) = p$ and $P(\Pi = \Pi^d) = q = 1 - p$, the equation (1.12) just tells that Π is the discounted expectation of this random variable under the probability $(p, 1-p)$, called *risk neutral probability :* we will come back on this at chapter 3. This probability is well defined by (1.13) as a function of u, d and r (or $R = e^{r\rho t}$) but it can also be defined as the unique p such that :

$$S_t = e^{-r\delta t}(pS_t u + (1-p)S_t d), \tag{1.14}$$

which means that p is the (unique) probability such that at any time t the value of the underlying asset is the discounted expectation of its time $t + \delta t$ value. This is why p is also called the *martingale probability.*

This risk neutral probability allows to consider S_t for all $t \in \mathbb{T}$ as a random variable taking the $i + 1$ values $\{S_0 u^j d^{i-j}, j = 0, \ldots, i\}$ with probabilities

$$P(S_t = S_0 u^j d^{i-j}) = \begin{pmatrix} i \\ j \end{pmatrix} p^j (1-p)^{i-j}. \tag{1.15}$$

Thus the following result should now be not surprising :

Proposition 1.1 (fundamental formula) *In a CRR model, the price Π_0 of an option $(T, \varphi(S_T))$ is given by*

$$\Pi_0 = e^{-rT}\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix} p^j (1-p)^{n-j}\varphi(S_0 u^j d^{n-j}) =: e^{-rT}\mathbb{E}^*(\varphi(S_T)) \tag{1.16}$$

where $\mathbb{E}^*$ denotes the expectation with respect to the risk neutral probability $\mathbb{P}^*$.

In other words, the premium of such an option is the discounted expectation, under the risk neutral probability, of its payoff. The formula (1.16) is called the *fundamental formula* for option pricing. The next section is devoted to set up more precisely a probabilistic framework for this construction.

Remark: Observe that the fundamental formula gives the option price at date $t = 0$ and it is easy to generalize it in order to obtain the option price at any date $t \in \mathbb{T}$. But this formula does not give directly the two components of the hedging portfolio. They are given by (1.11) : notice that the number of units of the underlying asset to use in the portfolio, called the *delta* of the portfolio or the *hedge ratio*, is, at date t, the difference in the value of the option at date $t + \delta t$ divided by the difference in the price of the underlying asset at the same date. It looks like the (mathematical) derivative of the option price with respect to the underlying asset price.

Exercise 1.1 We consider a european Put option on a risky asset S_t of initial value $S_0 = 100$ that may wether increase of 10% or diminish of 5% at each time period δt. One has acces to a non-risky asset the value of which increases of 2% at each time period. Determine the hedging portfolio (with α_t of risky asset and β_t of non risky asset at time t, thus bought at time $t - \delta t$) and compute its (initial) price in the two following cases :

1. The exercise date is after one time period ($T = \delta t$) and the exercise price is $K = 100$.

2. The exercise date is after two time periods ($T = 2\delta t$) and the exercise price is $K = 100$.

1.2 En route towards the stochastic approach

In the previous section we have adopted an approach of pricing and hedging of European options with payoff function φ using some elementary linear algebra and geometric understanding of the adopted model for the behaviour of the underlying asset and the hedging strategy. This leads to the price Π_0 given by proposition 1.1

$$\Pi_0 = e^{-rt}\mathbb{E}(\varphi(S_T)) \text{ , with } S_T = S_0 u^J d^{-n-J}.$$

where J is the number of "up" movements of the stock. As a matter of fact, J is a random variable on the the set Ω off all trajectories, defined on $\mathbb{T} := [0..T]_{\delta t} := \{0, \delta t, 2\delta t, \dots, n\delta t\}$, with $\delta t = T/n$, and the law of J being a binomial law $\mathcal{B}(n,p)$, with $p = \frac{e^{r\delta t}-d}{u-d}$. We'll detail this going a step further in the probabilist approach, taking also into account the *dynamic* of the asset S_t : such a "dynamic random variable" is called a *stochastic process* (or *random walk*). Here the most natural way to construct it is to consider J as the sum of n r.v. $(\delta J_i)_{i=1..n}$, where $\delta J_i(\omega)$ is equal to 1 or 0 according to the fact that the i-th change of the stock on the trajectory $\omega \in \Omega$ has an "up" or a "down" movement. So J is the n-th and last step of the random walk $(J_i)_{i=0..n}$ defined by $J_0 = 0$ and by induction $J_i = J_{i-1} + \delta J_i$. In terms of the stock price, the stochastic process is defined by S_0, a given non-random value, and by induction

$$S_t = S_{t-\delta t} u^{\delta J_i} d^{1-\delta J_i} = S_{t-\delta t}(u/d)^{\delta J_i} d \text{ , } t = i\delta t = iT/n, \tag{1.17}$$

thus, by induction on i,

$$S_t = S_0 u^{J_{i-1}+\delta J_i} d^{i-1-J_{i-1}+1-\delta J_i} = S_0 u^{J_i} d^{i-J_i}.$$

Let's consider also the dynamic of the price of the option ; we take from the elementary linear-geometric approach above that, at time $t = i\delta t$, the price Π_t of the hedging portfolio is a function of S_t so $\Pi_t = \phi(t, S_t)$ with of course $\phi(T, S_T) = \varphi(S_T)$, and, from (1.12), we have

$$\begin{aligned} \Pi_t &= p\phi(t+\delta t, S_t u) + (1-p)\phi(t+\delta t, S_t d) \text{ , with } p := \tfrac{e^{r\delta t}-d}{u-d} \text{ as in (1.13)}, && (1.18)\\ &= \mathbb{E}(\phi(t+\delta t, S_t u^{\delta J_{i+1}} d^{1-\delta J_{i+1}}) \mid S_t) = \mathbb{E}(\phi(t+\delta t, S_{t+\delta t}) \mid S_t), && (1.19) \end{aligned}$$

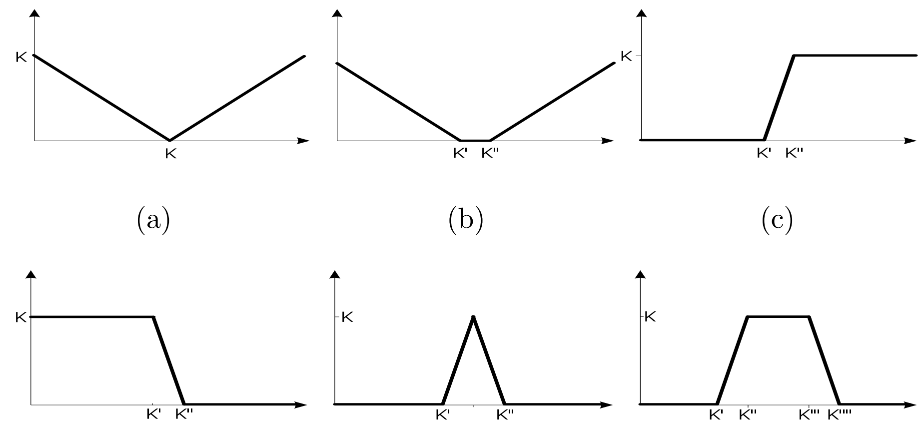
provided we assume S_t independent of δJ_{i+1}. This will be provided by the fact that *we just assume all de r.v.* $(\delta J_i)_{i=1..n}$ *to be independent* (and thus each J_i is a binomial r.v. : $J_i \rightsquigarrow \mathcal{B}(p,i)$) : this just means that knowing $\delta J_1, \dots, \delta J_i$ gives no hint on what δJ_{i+1} will be.

The nice way now to understand the fundamental formula of proposition 1.1 using stochastic processes theory is to understand (1.17) as the solution of a stochastic difference equation and to push (1.19) still a bit further to turn it into a conditional expectation with respect to the algebra $\mathcal{F}_t^S$ modeling the information available at time t from the observed prices $(S_0, S_{\delta t}, \dots, S_t)$.This is what we shall set up in the next chapter, at theorem 2.1.

1.3 Model-free properties

1.3.1 Arbitrage

An arbitrage opportunity is the chance to buy and sell at the same time various assets to build up a so-called *arbitrage portfolio* in such a way that :



(a) (b) (c) (d) (e) (f)

Figure 1.5: Payoff functions of some standard options : (a) straddel, (b) strangel, (c) bull spread, (d) bear spread, (e) butterfly spread, (f) condor.

- this portfolio costs you nothing or even it gives you some money in as you (short[1] -)sell for more than you buy
- the resulting position is not risky, as the portfolio will have never negative value (even if you will have to provide at some time in the future the assets that you short-sold)
- there is a chance that at some date in the future yous portfolio will have positive value, in which case you will unbuckle your position, selling your portfolio at its positive value, thus making money.

Obviously, such an arbitrage looks like the philosopher's stone[2] and does sometimes exist, but only for a short time (about 30") as banks do hire "arbitragers" to take advantage of it to the benefit of the bank.

Up to the above mentionned exceptional cases, arbitrages can be considered as non existing, and provide usefull modeling tools, as a model exhibiting arbitrages are irrealistic. From the mathematical finance point of view, the absence of arbitrage provides bridges between real life and mathematical reasoning. We will consider below two examples of observations that are *model free*, which means that any mathematical model should satisfy them. We than will come back again on absence of arbitrage at chapter 4.

1.3.2 The butterfly spread

A butterfly spread is a european vanilla option with payoff function as on figure 1.5(e). It can be built with three call-option with same exercise date T, and three different strike-prices $K' < K < K$", thus with payoff functions $\varphi_{K'}(S_T) = (S_T - K')^+$, $\varphi_K(S_T) = (S_T - K)^+$, and $\varphi_{K''}(S_T) = (S_T - K'')^+$. It is easy to see that, choosing a and b such that

$$a = \frac{K'' - K'}{K'' - K} \text{ , and } b = \frac{K - K'}{K'' - K}, \tag{1.20}$$

the payoff φ of the butterfly spread is just

$$\varphi = \varphi_{K'} - a\varphi_K + b\varphi_{K''}. \tag{1.21}$$

[1]to short-sell an asset is to sell something you do not actually have, as you do not have to produce the asset immediately, and that you will buy later, is the hope to make a profit or the hedge an option. This is allowed.

[2]that was supposed to turn lead into gold

Here comes a typical arbitrage argument : as the payoff of the butterfly spread is positive in any case, then the prices C'_t, C_t, and C''_t of call-options with exercise date T, and three strike-prices K', K, and K'' should satify at any time $t < T$ the relation

$$C'_t - aC_t + bC''_t > 0. \tag{1.22}$$

Indeed, would the prices on the market at some time t^* be such that $\Pi_{t^*} := C'_{t^*} - aC_{t^*} + bC''_{t^*} \leq 0$, you would build up at a non-positive price a portfolio of the corresponding quantities of the three types of options ; so you pay nothing and even get a (first) free lunch of $-\Pi_{t^*}$, and, if you are lucky S will end up at S_T, with $K' < S_T < K''$, so your butterfly spread (portfolio) will have positive value, providing a (second) free lunch of $\varphi(S_T)$. If not, it ends up with zero value so that in any case you lose no money. So, relation (1.22) is a no-arbitrage relation between any triple of call-option prices that should be true any time (thus in any reasonable (i.e. no-arbitrage) model).

1.3.3 The Call-Put relations

Certainly the most well-known arbitrage relation is the Call-Put relation. It results from the fact that $(S-K)^+ - (K-S)^+ = S - K$ for all S and K. If we denote by C_t and P_t the price at time $t \leq T$ of a call-option and and a put-option with exercise date T and strike-price K, and if at time t we can borrow $e^{-r(T-t)}$ for an euro to be payed at time T we have "by arbitrage"

$$C_t - P_t = S_t - e^{-r(T-t)}K. \tag{1.23}$$

Indeed, if for example at some time t^* you observe that $C_t - P_t > S_t - e^{-r(T-t)}K$, then you would (short-)sell C_t, buy S_t, and P_t and borrow $e^{-r(T-t)}K$, so ending up, by assumption with some cash for a lunch. At time T the value of your portfolio will be $S_T - K - (C_T - P_T) = S_T - K - ((S_T - K)^+ - (K - S_T)^+) = 0$. So your lunch was free. Of course, if $C_t - P_t < S_t - e^{-r(T-t)}K$ you would just take the converse strategy.

Exercise 1.2 *Denote by* $\varphi_K^C(s) := (S-K)^+$ *and* $\varphi_K^P(s) := (K-S)^+$ *the payoff functions of a Call and of a Put option with strike price* K. *Then the payoff of a* straddel *option is just* $\varphi_K^C + \varphi_K^P$ *; express similarly the payoff function of the other examples of options given in figure 1.5*

Chapter 2

European options

2.1 What is a general option ?

A general european option on a stock S with exercice date T is a derivative the payoff Π_T of which is to be payed at time T but does not only depend on the value S_T of S at time T but, possibly, on all values S_t of S for $0 \le t \le T$. The study of the question of pricing (and hedging) an european option in a stochastic model $(S_t)_{t\ge 0}$ will thus involve naturally the algebra $\mathcal{F}_T^S$ generated by all the random variables S_t, $0 \le t \le T$, and, more generally, any algebra $\mathcal{F}_t^S$ generated by all the r.v. S_s, $0 \le s \le t$. In this way we form the *filtration* $(\mathcal{F}_t^S)_{0\le t\le T}$, which just means that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for any $0 \le s \le t$. As we deal with a model in discrete time ($t \in [0..T]_{\delta t}$), we have $\mathcal{F}_t^S = \sigma(S_{\delta t}, \dots, S_t)$ so any r.v. X is $\mathcal{F}_t$-measurable if and only if $X = f(S_{\delta t}, \dots, S_t)$ for some (deterministic) function f. As in our model $S_{k\delta t}$ has only a finite number of values, the algebra $\mathcal{F}_t$ is generated by its atoms

$$\overline{\omega}_t := \{\omega' \in \Omega \mid S_{\delta t} = s_{\delta t}, \dots, S_t = s_t\} \text{ , with } s_{k\delta t} = S_0 u^{j_k} d^{n-j_k} \text{ , } 0 \le j_k := J_k(\omega) \le k.$$

In other words, in our model, a european option is a derivative that pays, at time T, an amount $\Pi_T(\omega)$ depending on the trajectory $\omega \in \Omega$ that is of type $\Pi_T(\omega) = \pi(T, s_{\delta t}, \dots, s_T)$ for $s_{k\delta t} = S_{k\delta t}(\omega)$, for some function $(s_{\delta t}, \dots, s_{n\delta t}) \mapsto \pi(T, s_{\delta t}, \dots, s_{n\delta t})$. We let here the deterministic function π also depend on $t \in [0..T]_{\delta t}$ to allow to write the value $\pi(k\delta t, s_{\delta t}, \dots, s_{k\delta t}, 0, \dots, 0) =: \pi(k\delta t, s_{\delta t}, \dots, s_{k\delta t})$ of the option at any time $t \in [0..T]_{\delta t}$, and in particular, at $t = 0$: the *premium* of the option.

2.2 Barrier options

Barrier options are good examples of european options that are traded, not on stock-exchanges, but "over the conter" (OTC), in finance bank. They are Call or Put options that would pay the usual payoff $(S-K)^+$ or $(K-S)^+$ but only if the stock assumes for some or all $t \in [0..T[_{\delta t}$ values larger or smaller to some value L decided in advance, called the barrier value. So we have Up barrier options and Down barrier options (according to requiering $S_t \ge L$ or $S_t < L$). We have (knock-)In options (that would pay only if there is some $t \in [0..T[$ such that S_t crosses the barrier L) and (knock-)Out option (that would pay only if S_t does not cross the barrier L – other wise the option would be knocked-out !–). Finally we have eight different barrier options : DIC,DIP,UIC,UIP, DOC,DOP,UOC,UOP.

In order to express the payoff of such an option, it is convenient to introduce the sets

$$\mathcal{D}_L(\omega) := \{t \in [0..T[\mid S_t(\omega) < L\} \text{ , } \omega \in \Omega,$$

and the r.v. $\tau_L \in [0..T]_{\delta t}$

$$\tau_L(\omega) = \begin{cases} \operatorname{Min} \mathcal{D}_L(\omega) & \text{if } \mathcal{D}_L(\omega) \ne \emptyset \\ T & \text{if } \mathcal{D}_L(\omega) = \emptyset. \end{cases}$$

So, for example, the payoff of a DIC (Down and In Call) with strike K and barrier L is

$$\Pi_T = DIC_T := (S_T - K)^+ \mathbb{I}_{\{\tau_L < T\}}.$$

Just to mention, the r.v. τ_L is a *stopping-time for the filtration* $(\mathcal{F}_t^S)_{t\in[0..T]}$ as any event $\{\tau_L \leq t\}$ is $\mathcal{F}_t^S$-measurable : at any time t, knowing $\mathcal{F}_t^S = \sigma(S_{\delta t}, \ldots, S_t)$ allows to decide if the stopping-time $\tau_L(\omega)$ "has come", i.e. $\tau_L(\omega) \leq t$.

2.3 Pricing a european option

In our binary model we can easily adapt the hedging/pricing strategy explained for vanilla options in order to get a hedging strategy for exotic options. Recall, this just works by backward induction : at time T the value is known from the payoff $\Pi_T := \pi(T, S_{\delta t}, \ldots, S_{n\delta t})$ for the considered option. Now assume $\pi(t+\delta t, s_{\delta t}, \ldots, s_{t+\delta t})$ to be already known. We claim that the value of the hedging portfolio at time t (depending on $s_{\delta t} = S_{\delta t}(\omega)$,..., $s_t = S_t(\omega)$), satisfies

$$R\pi(t, s_{\delta t}, \ldots, s_t) = p\pi(t+\delta t, s_{\delta t}, \ldots, s_{t+\delta t}u) + (1-p)\pi(t+\delta t, s_{\delta t}, \ldots, s_{t+\delta t}), \tag{2.1}$$

for $R := e^{r\delta t}$ and $p := \frac{R-d}{u-d}$. Indeed, the same reasoning holds : at time t we observe that $S_{\delta t} = s_{\delta t}$,...,$S_t(\omega) = s_t$, and are facing two possible values for $S_{t+\delta t}(\omega)$ and $\Pi_{t+\delta t}(\omega)$, namely $s_t u$ and $s_t d$ for $S_{t+\delta t}(\omega)$, and $\pi(t+\delta t, s_{\delta t}, \ldots, s_t, s_t u)$ and $\pi(t+\delta t, s_{\delta t}, \ldots, s_t, s_t d)$ for $\Pi_{t+\delta t}(\omega)$, so we compute the number x of stocks and y of bonds to hold in our hedging portfolio in order to have a portfolio that has exactly the right value in any of both possible issues, and we already explained at chapter 1 that the present value of this portfolio is given by (2.1).

Now we want to fit this nice elementary tric into a conditionnal expectation formalism, to express nicely this value $\Pi_t := \pi(t, S_{\delta t}, \ldots, S_t)$.

Theorem 2.1 *Let Π_t be the $\mathcal{F}_t^S$-measurable payoff of any european option. This value Π_t at time t of the hedging portfolio in the CRR model is given by*

$$\Pi_t = e^{-r(T-t)}\mathbb{E}(\Pi_T \mid \mathcal{F}_t^S). \tag{2.2}$$

In particular, the premium of the option is equal to $\Pi_0 = e^{-rT}\mathbb{E}(\Pi_T)$.

Proof: Let us first use (2.1) to show the following partial formula :

$$\Pi_t = \frac{1}{R}\mathbb{E}(\Pi_{t+\delta t} \mid \mathcal{F}_t^S). \tag{2.3}$$

For any $\omega \in \Omega$, let $s_{k\delta t} := S_{k\delta t}(\omega)$, $k = 1..n$. Recall that $\overline{\omega}_t$ denotes the atom of ω in the algebra $\mathcal{F}_t^S$,

$$\overline{\omega}_t = \{\omega' \in \Omega \mid S_{\delta t}(\omega') = s_{\delta t}, \ldots, S_t(\omega') = s_t\}.$$

By definition of the conditionnal expectation we have

$$\begin{aligned}
& \mathbb{E}(\Pi_{t+\delta t} \mid \mathcal{F}_t^S)(\omega) \\
= \; & \mathbb{E}(\Pi_{t+\delta t}\mathbb{I}_{\overline{\omega}_t})\frac{1}{\mathbb{P}(\overline{\omega}_t} \\
= \; & \mathbb{E}\left(\pi\left(t+\delta t, S_{\delta t}, \ldots, S_t, S_t d\left(\frac{u}{d}\right)^{\delta J_{t+\delta t}}\right)\mathbb{I}_{\{S_\delta t = s_\delta t\}}\cdots\mathbb{I}_{\{S_t=s_t\}}\right)\frac{1}{\mathbb{E}(\mathbb{I}_{\{S_{\delta t}=s_{\delta t}\}}\cdots\mathbb{I}_{\{S_t=s_t\}})} \\
= \; & \mathbb{E}\left(\pi\left(t+\delta t, s_{\delta t}, \ldots, s_t, s_t d\left(\frac{u}{d}\right)^{\delta J_{t+\delta t}}\right)\mathbb{I}_{\{S_\delta t = s_\delta t\}}\cdots\mathbb{I}_{\{S_t=s_t\}}\right)\frac{1}{\mathbb{E}(\mathbb{I}_{\{S_{\delta t}=s_{\delta t}\}}\cdots\mathbb{I}_{\{S_t=s_t\}})} \\
= \; & \mathbb{E}\left(\pi\left(t+\delta t, s_{\delta t}, \ldots, s_t, s_t d\left(\frac{u}{d}\right)^{\delta J_{t+\delta t}}\right)\right)\frac{\mathbb{E}(\mathbb{I}_{\{S_{\delta t}=s_{\delta t}\}}\cdots\mathbb{I}_{\{S_t=s_t\}})}{\mathbb{E}(\mathbb{I}_{\{S_{\delta t}=s_{\delta t}\}}\cdots\mathbb{I}_{\{S_t=s_t\}})} \text{ , by independance,} \\
= \; & p\pi(t+\delta t, s_{\delta t}, \ldots, s_t, s_t u) + (1-p)\pi(t+\delta t, s_{\delta t}, \ldots, s_t, s_t d) \text{ , as } \delta J_{t+\delta t} \rightsquigarrow \mathcal{B}(1,p), \\
= \; & R\,\pi(t, s_{\delta t}, \ldots, s_t) \text{ , by (2.1),} \\
= \; & R\,\pi(t, S_{\delta t}, \ldots, S_t)(\omega) = R\,\Pi_t(\omega).
\end{aligned}$$

Now we can show more generally, by induction on l, that

$$\Pi_t = \frac{1}{R^l}\mathbb{E}(\Pi_{t+l\delta t} \mid \mathcal{F}_t^S). \tag{2.4}$$

Indeed, formula (2.3) shows that (2.4) is true for $l = 1$; assume by induction that (2.4) is true ; then, replacing t by $t + l\delta t$ in formula (2.3),

$$\begin{aligned}
\Pi_t &= \frac{1}{R^l}\mathbb{E}(\Pi_{t+l\delta t} \mid \mathcal{F}_t^S) \text{ , by assumption} \\
&= \frac{1}{R^l}\mathbb{E}\left(\frac{1}{R}\mathbb{E}(\Pi_{t+l\delta t+\delta t} \mid \mathcal{F}_{t+l\delta t}^S) \mid \mathcal{F}_t^S\right) \text{ , by formula (2.3),} \\
&= \frac{1}{R^{l+1}}\mathbb{E}(\mathbb{E}(\Pi_{t+(l+1)\delta t} \mid \mathcal{F}_{t+l\delta t}^S) \mid \mathcal{F}_t^S) \\
&= \frac{1}{R^{l+1}}\mathbb{E}(\Pi_{t+(l+1)\delta t} \mid \mathcal{F}_t^S) \text{ , as } \mathcal{F}_t^S \subseteq \mathcal{F}_{t+l\delta t}^S,
\end{aligned}$$

which shows (2.4) for $l + 1$. Now, choosing l such that $t + l\delta t = T$, one gets

$$\Pi_t = \frac{1}{R^l}\mathbb{E}(\Pi_{t+l\delta t} \mid \mathcal{F}_t^S) = e^{-r(T-t)}\mathbb{E}(\Pi_T \mid \mathcal{F}_t^S).$$

□

Exercise 2.1 *Let* $(\Omega, \mathbb{P}, \mathbb{F})$ *be a filtered probability space (i.e.* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0..T]_{\delta t}}$ *is a fitration on* Ω*). A r.v.* τ *on* Ω *is a* stopping-time *for* $\mathbb{F}$ *if and only if for any* $t_0 \in [0..T]_{\delta t}$ *the event* $\{\tau \le t_0\} \in \mathcal{F}_{t_0}$ *or equivalently the r.v.* $\mathbb{I}_{\{\tau\le t_0\}}$ *is* $\mathcal{F}_{t_0}$*-measurable (so, at any time* t_0 *you know if "the stopping time* τ *has already come"). Assume* Ω *is finite and let* $(S_t)_{t\in[0..T]_{\delta t}}$ *be a stochastic processes. Let* $A \subseteq \mathbb{R}$ *be any subset and, for any* $t \in [0..T]_{\delta t}$, $T_A^S(\omega) = \{t \in [0..T]_{\delta t} \mid S_t(\omega) \in A\}$. *Define* τ_A *to be the r.v. defined by* $\tau_A(\omega) = Min T_A^S(\omega)$ *if* $T_A^S(\omega) \neq \emptyset$ *and by* $\tau_A(\omega) = T + 1$ *if* $T_A^S(\omega) = \emptyset$

1. *Show that* $S_t^{-1}(A)(:= \{\omega \in \Omega \mid S_t(\omega) \in A\}) \in \mathcal{F}_t$
2. *Show that* $\{\tau_A \le t_0\}(:= \{\omega \in \Omega \mid \tau_A(\omega) \le t_0\}) = \bigcup_{t\in[0..t_0]_{\delta t}} S_t^{-1}(A)$
3. *Show that* τ_A *is a stopping-time for* $\mathbb{F}(S)$.
4. *For any of the barrier options DIC,DIP,UIC,UIP, DOC,DOP,UOC,UOP, define* φ *and* A *such that its pay-off function is given by* $\varphi(S_T)\mathbb{I}_{\{\tau_A\le T\}}$.

Chapter 3

Profit'n Loss

3.0 Actualization

In the previous chapter we have shown the fundamental theorem for the pricing of derivatives on a random asset S, namely that $\Pi_t = e^{-r(T-t)}\mathbb{E}(\Pi_T \mid \mathcal{F}_t^S)$. This formula is nicely compact, but there is a trick that makes it even more compact, namely

$$\tilde{\Pi}_t = \mathbb{E}(\tilde{\Pi}_T \mid \mathcal{F}_t^S).$$

This trick consists in defining $\tilde{\Pi}_t := e^{-rt}\Pi_t$ or, more generally :

Definition: Given a market with constant rate r, the *present value* $(\tilde{X}_t)$ of any process (X_t) is defined by

$$\tilde{X}_t = e^{-rt}X_t.$$

Exercise: Show that, indeed, $\tilde{\Pi}_t = \mathbb{E}(\tilde{\Pi}_T \mid \mathcal{F}_t^S)$.

If one interprets X_t as the amount of Philippine Pesos (PHP) that will be available to you at time t, if you need to borrow some money today, at time 0, with interest rate r, than you will get $\tilde{X}_t = e^{-rt}X_t$, so that the amount $e^{rt}\tilde{X}_t$ you will have to pay back when X_t will be available is precisely equal to that amount : this is just financial math.

It turns out, when we want to go to mathematical finance and to stochastic models, that it is very convenient to deal with processes (M_t) such that $M_t = \mathbb{E}(M_T \mid \mathcal{F}_t^S)$ provided $t \leq T$, the so-called martingale processes. The trick above is called *actualization ;* here we presented actualization at present time ; we could as well actualize at time T, setting $\hat{X}_t = e^{r(T-t)}X_t$, as $\hat{X}_t$ is the amount you will have at time T if you invest X_t riskless at rate r ; an example of such $\hat{X}_t$ is the so-called *zero-coupon bond* B_t that is such that $\hat{B}_t = 1$ at any time $t \leq T$, as it will pay one (or more likely one million) PHP at time T (and no *coupon* in the interval, which explains its name).

In the context of constant interest rates it is very easy to deduce the theory for (X_t) from a stochastic theory valid for $(\tilde{X}_t)$, and, as this latter theory involves martingales, this is what the mathematician should do (and we will do !). In order to avoid to litter the computations with boring tildas $\tilde{}$, we will just assume $r = 0$: this does not mean that the results can't help for markets with constant non-zero interest rates ; it just means that when applying a result in a real market with fixed $r > 0$, you should first add tildas in the theorems you want to apply and that assumed $r = 0$ and find out (easily) what the result becomes without the tildas, replacing all $\tilde{X}_t$ by $e^{-rt}X_t$.

3.1 Martingales with respect to $\mathbb{F}(S)$

Recall that we consider the CRR model for which $S_t = S_0 u^{J_t} d^{k-J_t}$, $t = k\delta t \in [0..T]_{\delta t}$, with $J_{k\delta t} = \sum_{i=1..k} \delta J_{i\delta t}$, where the $(\delta J_{i\delta t})_{i=1..n}$ are independant Bernoulli r.v., $\delta J_{i\delta t} \rightsquigarrow \mathcal{B}(1,p)$, with $p = \frac{R-d}{u-d}$, $R = e^{r\delta t}$, $u > R > d > 0$; so all $J_{k\delta t}$ are Binomial r.v., $J_{k\delta t} \rightsquigarrow \mathcal{B}(k,p)$, of course not independent. Denote by $(\Omega, \mathbb{P})$ any **finite** probability space on which such $(\delta J_{i\delta t})_{i=1..n}$ are defined. As explained in the previous section we will assume $r = 0$, so $R = 1$ and $u > 1 > d$.

Let $\mathcal{F}_t^S := \sigma(S_{\delta t}, S_{2\delta t}, \ldots, S_{k\delta t})$, $t = k\delta t$, and consider the filtration $\mathbb{F}(S) := (\mathcal{F}_t^S)_{t\in[0..T]_{\delta t}}$. It is easy to see that

$$\mathcal{F}_t^S = \mathcal{F}_t^J := \sigma(J_{\delta t}, J_{2\delta t}, \ldots, J_{k\delta t}) = \mathcal{F}_t^{\delta J} := \sigma(\delta J_{\delta t}, \delta J_{2\delta t}, \ldots, \delta J_{k\delta t}).$$

Define

$$\delta S_t := S_t - S_{t-\delta t} = S_{t-\delta t}(u^{\delta J_t} d^{1-\delta J_t} - 1) \ ;$$

of course we also have $\mathcal{F}_t^S = \sigma(\delta S_{\delta t}, \delta S_{2\delta t}, \ldots, \delta S_{k\delta t}) = \mathcal{F}_t^{\delta S}$. Now it is easy to see that

$$\mathbb{E}(S_t \mid \mathcal{F}_{t-\delta t}^S) = S_{t-\delta t}. \tag{3.1}$$

Indeed, observe that $S_{t-\delta t} \in \mathcal{F}_{t-\delta t}^S = \mathcal{F}_{t-\delta t}^{\delta J}$, so δJ_t is independent of $\mathcal{F}_{t-\delta t}^S$, thus

$\mathbb{E}(S_t \mid \mathcal{F}_{t-\delta t}^S) = \mathbb{E}(S_{t-\delta t} u^{\delta J_t} d^{1-\delta J_t} \mid \mathcal{F}_{t-\delta t}^S) = S_{t-\delta t}\mathbb{E}(u^{\delta J_t} d^{1-\delta J_t} \mid \mathcal{F}_{t-\delta t}^{\delta J}) = S_{t-\delta t}\mathbb{E}(u^{\delta J_t} d^{1-\delta J_t}) = S_{t-\delta t}$, as

$$\mathbb{E}(u^{\delta J_t} d^{1-\delta J_t}) = pu + (1-p)d = \frac{1-d}{u-d}u + \frac{u-1}{u-d}d = 1.$$

Formula (3.1) shows that the process $(S_t)_{t\in[0..T]_{\delta t}}$ has an interesting property : it is a *martingale :*

Definition: Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0..T]}$ be a filtration on a (finite) probability space $(\Omega, \mathbb{P})$. A process $(M_t)_{t\in[0..T]}$ is called a $(\mathbb{P}, \mathbb{F})$-martingale if and only if for any s and t in $[0..T]$, if $s \le t$ then

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s. \tag{3.2}$$

Proposition 3.1 *Let $M = (M_t)_{t\in[0..T]}$ be a process on the finite probability space $(\Omega, \mathbb{P})$, and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0..T]}$ be a filtration on Ω. The following properties are equivalent :*

1. *M is $\mathbb{F}$-adapted and $\mathbb{E}(\delta M_{t+\delta t} \mid \mathcal{F}_t) = 0$ for any $t \in [0..T)$,*
2. *$\mathbb{E}(M_{t+\delta t} \mid \mathcal{F}_t) = M_t$ for any $t \in [0..T)$,*
3. *M is a $(\mathbb{P}, \mathbb{F})$-martingale.*

Proof: By (3.2), obviously 3. implies 2.. Assume 2. so, $M_t \in \mathcal{F}_t$, and thus $\mathbb{E}(M_t \mid \mathcal{F}_t) = M_t$, so

$$\mathbb{E}(\delta M_{t+\delta t} \mid \mathcal{F}_t) = \mathbb{E}(M_{t+\delta t} - M_t \mid \mathcal{F}_t) = \mathbb{E}(M_{t+\delta t} \mid \mathcal{F}_t) - \mathbb{E}(M_t \mid \mathcal{F}_t) = M_t - M_t = 0,$$

so 1. holds.

Assume 1., and let us show

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s \text{ for } t = s + k\delta t, \tag{3.3}$$

by induction on k. As M is adapted, $\mathbb{E}(M_t \mid \mathcal{F}_t) = M_t$, and $0 = \mathbb{E}(\delta M_{t+\delta t} \mid \mathcal{F}_t) = \mathbb{E}(M_{t+\delta t} \mid \mathcal{F}_t) - M_t$, so $\mathbb{E}(M_{t+\delta t} \mid \mathcal{F}_t) = M_t$. So (3.3) for $k = 0$ and $k = 1$. Let $k > 1$ and assume (3.2) holds for any s' and t such that $t = s' + (k-1)\delta t$. Now let $t = s + k\delta t$; we have

$$\begin{aligned}
\mathbb{E}(M_t \mid \mathcal{F}_s) &= \mathbb{E}(M_{s+\delta t+(k-1)\delta t} \mid \mathcal{F}_s) \\
&= \mathbb{E}(\mathbb{E}(M_{s+\delta t+(k-1)\delta t} \mid \mathcal{F}_{s+\delta t}) \mid \mathcal{F}_s) \\
&= \mathbb{E}(M_{s+\delta t} \mid \mathcal{F}_s) \text{ , by assumption,} \\
&= M_s \text{ , as (3.3) is true for } k = 1.
\end{aligned}$$

□

Examples:

- By (3.1), applying the previous proposition we see that, as we assumed $r = 0$, the CRR model $S := (S_t)_{t\in[0..T]_{\delta t}}$ is a $(\mathbb{P}, \mathbb{F}(S))$-martingale.
- From the fundamental theory of option pricing we see that in the CRR model, if $r = 0$ the price of any european (vanilla or exotic) option is a $(\mathbb{P}, \mathbb{F}(S))$-martingale. Indeed, $\Pi_t = \mathbb{E}(\Pi_T \mid \mathcal{F}_t^S)$ as $r = 0$ thus, for any $s \leq t$ in $[0..T]$, as $\mathcal{F}_s^S \subseteq \mathcal{F}_t^S$, we have

$$\mathbb{E}(\Pi_t \mid \mathcal{F}_s^S) = \mathbb{E}(\mathbb{E}(\Pi_T \mid \mathcal{F}_t^S) \mid \mathcal{F}_s^S) = \mathbb{E}(\Pi_T \mid \mathcal{F}_s^S) = \Pi_s.$$

Actually, we have just shown the following general result :

Proposition 3.2 *Let X be any r.v. on a finite probability space $(\Omega, \mathbb{P})$ and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0..T]}$ be any filtration on Ω. The conditional expectation process with respect to $\mathbb{F}$*

$$M(X,T) = \left(M_t^{X,T}\right)_{t\in[0..T]} := (\mathbb{E}(X \mid \mathcal{F}_t))_{t\in[0..T]} \tag{3.4}$$

is a $(\mathbb{P}, \mathbb{F})$-martingale.

The next section will provide one more important example of $(\mathbb{P}, \mathbb{F})$-martingale.

3.2 Profit and loss of a predictable strategy

Recall that, when determining the hedge of an option, the quantity α_t of stock held in the hedge portfolio was arranged in advance, at time $t - \delta t$, when only $S_{t-\delta t}$ was known. Such a process $(\alpha_t)_{t\in(0..T]_{\delta t}}$ is called $\mathbb{F}(S)$-predictable. More generally :

Definition: Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0..T]}$ be any filtration on Ω. An $\mathbb{F}$-adapted process $(\alpha_t)_{t\in(0..T]_{\delta t}}$ is called $\mathbb{F}$-predictable if and only if for any $t \in (0..T]_{\delta t}$, α_t is $\mathcal{F}_{t-\delta t}$-measurable.

Definition: Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0..T]}$ be any filtration on Ω. Let $\alpha := (\alpha_t)_{t\in(0..T]_{\delta t}}$ be an $\mathbb{F}$-predictable process. The Profit-and-Loss of (the strategy) α on (the changes of) S is the process $(P\&L_t^S(\alpha))_{t\in(0..T]_{\delta t}}$ defined by

$$P\&L_t^S(\alpha) := \sum_{s\in(0..t]_{\delta t}} \alpha_s \delta S_s \text{ (where } \delta S_s = S_s - S_{s-\delta t}\text{ .)} \tag{3.5}$$

This terminology is quite natural : at time $s - \delta t$ you take a position of α_s stocks at the price $S_{s-\delta t}$; at time s the price S has changed of $\delta S_s = S_s - S_{s-\delta t}$, so you made a profit (or loss) of $\alpha_s \delta S_s$, and you take a new position $\alpha_{s+\delta t}$ for the next time step (with only the information $\mathcal{F}_s^S$ available). So $P\&L_t^S(\alpha)$ is just your total profit and loss up to time t. Actually, $S_t - S_0$ is just $P\&L_t^S(1)$, the profit and loss of the *buy-and-hold strategy.*

From the mathematical point of vue, $(P\&L_t^S(\alpha))_{t\in(0..T]_{\delta t}}$ is the *Itô stochastic integral*

$$\int_0^T \hat{\alpha}_s \mathbb{I}_{\{s\leq t\}} d\mu_s \text{ , with } \mu_s := \sum_{j=0}^{k(s)} \binom{n}{j} p^j (1-p)^{k(s)-j} \delta_{S_0 u^j d^{k(s)-j}} \text{ and}$$

$$\hat{\alpha}(s) = \alpha_{k(s)} \text{ for } s \in [k(s)\delta t, (k(s)+1)\delta t),\ k(s) \in \mathbb{N}.$$

of the $\mathbb{F}(S)$-predictable process $(\alpha_t)_{t\in(0..T]_{\delta t}}$ against the changes of the $(\mathbb{P}, \mathbb{F}(S))$-martingale $S = (S_t)_{t\in[0..T]_{\delta t}}$. Defining a stochastic integral becomes much more sophisticated when an infinite Ω is necessary, but the fundamental idea stays exactly the same in the case of the so-called Itô integral : you just have to perform some elegant measure-theory and Hilbert-space acrobatics, and thus take care of some integrability conditions. Anyway, we shall adopt the notation

$$P\&L_t^S(\alpha) =: \int_0^t \alpha_s dS_s. \tag{3.6}$$

This elementary stochatic integral is perfectly well adapted to the modeling in finance. Its most (and serious) draw back comes from the fact that it is not well suited for "changes of unknown" of the kind $\beta_t := f(t, \alpha_t)$, for which the Itô integral benefits of the famous "Itô formula".

As for the Itô integral, the profit and loss in a CRR model has the interesting property to be a martingale :

Proposition 3.3 *For any $\mathbb{F}(S)$-predictable strategy $(\alpha_t)_{t\in(0..T]_{\delta t}}$, the profit and loss process $(P\&L_t^S(\alpha))_{t\in(0..T]_{\delta t}}$ is a $(\mathbb{P}, \mathbb{F}(S))$-martingale.*

Proof: $P\&L_t^S(\alpha) := \sum_{s\in(0..t]_{\delta t}} \alpha_s \delta S_s$ is obviously $\mathcal{F}_t^S$-adapted. We have

$$\delta(P\&L(\alpha))_{t+\delta t}^S = \sum_{s\in(0..t+\delta t]_{\delta t}} \alpha_s \delta S_s - \sum_{s\in(0..t]_{\delta t}} \alpha_s \delta S_s = \alpha_{t+\delta t} S_{t+\delta t}.$$

that is obviously $\mathbb{F}(S)$-adapted. Now, as α is $\mathbb{F}(S)$-predictable, $\alpha_{t+\delta t} \in \mathcal{F}_t^S$, so

$$\mathbb{E}(\delta P\&L_{t+\delta t}^S(\alpha) \mid \mathcal{F}_t^S) = \mathbb{E}(\alpha_{t+\delta t}\delta S_{t+\delta t}(\alpha) \mid \mathcal{F}_t^S) = \alpha_{t+\delta t}\mathbb{E}(\delta S_{t+\delta t}(\alpha) \mid \mathcal{F}_t^S) = 0,$$

as S is a $(\mathbb{P}, \mathbb{F}(S))$-martingale. One concludes applying proposition 3.1. □

3.3 Martingales representation

In the previous section we saw how the integrand α of a stochastic integral can be interpreted as a predictable strategy and that a stochastic integral is necessarily a martingale. So a natural question is to wonder if a given martingale corresponds to the profit and loss of a predictable strategy. Let us put two definitions in order to state the problem conveniently, and prove a theorem providing hypothesis leading to a positive answer.

Definition: Let $\mathbb{F}(S)$ be the filtration of a $(\mathbb{P}, \mathbb{F}(S))$-martingale $S = (S_t)_{t\in[0..T]_{\delta t}}$, and let $M = (M_t)_{t\in[0..T]_{\delta t}}$ be any $(\mathbb{P}, \mathbb{F}(S))$-martingale. An *$S$-representation of M* is an $\mathbb{F}(S)$-predictable process $\alpha = (\alpha_t)_{t\in(0..T]_{\delta t}}$ such that, for any t,

$$M_t - M_0 = P\&L_t^S(\alpha) \left(= \int_0^t \alpha_s dS_s \right). \tag{3.7}$$

Definition: Let $S = (S_t)_{t\in[0..T]_{\delta t}}$ be a $(\mathbb{P}, \mathbb{F}(S))$-martingale. We shall say that it has the *martingale representation property* (MRP) if and only if any $(\mathbb{P}, \mathbb{F}(S))$-martingale $M = (M_t)_{t\in[0..T]_{\delta t}}$ admits a predictable S-representation.

Theorem 3.4 *Any CRR model $S = (S_t)_{t\in[0..T]_{\delta t}}$ has the MRP.*

Proof: We will define the strategy $\alpha = (\alpha_t)_{t\in(0..T]_{\delta t}} = (\alpha_{k\delta t})_{k=1..n}$ by induction on k. Let $\alpha_0 = 0$ and assume α_s already defined for any $s \leq (k-1)\delta t$ in $[0..T]_{\delta t}$. So $M_{(k-1)\delta t} - M_0 = \sum_{s\in(0..(k-1)\delta t]} \alpha_s \delta S_s$. We need to choose $\alpha_{k\delta t} \in \mathcal{F}_{(k-1)\delta t}^S$ such that

$$\alpha_{k\delta t}\delta S_{k\delta t} = \delta M_{k\delta t} = m(k\delta t, S_{\delta t}, \dots, S_{(k-1)\delta t}) \tag{3.8}$$

for some (deterministic) function $m(t, s_1, \dots, s_{k-1})$. We need a lemma :

Lemma 3.5 *For $k\delta t = t$ and any deterministic fonction $f\ : \mathbb{R}^k \longrightarrow \mathbb{R}$ one has for any CRR model S*

$$\mathbb{E}(f(S_{\delta t}, \dots, S_{t-\delta t}, S_t) \mid \mathcal{F}_{t-\delta t}^S) = pf(S_{\delta t}, \dots, S_{t-\delta t}, S_{t-\delta t}u) + (1-p)f(S_{\delta t}, \dots, S_{t-\delta t}, S_{t-\delta t}d)$$

Proof: Recall that in any CRR model $S_t = S_{t-\delta t} U_t$, where $U_t := u^{\delta J_t} d^{1-\delta J_t} \in \{u, d\}$ is independent of $\mathcal{F}^S_{t-\delta t}$. If we define $X := f(S_{\delta t}, \ldots, S_{t-\delta t}, S_t)$ and $Y := pf(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} u) + (1 - p) f(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} d)$, we have to show that for any $A \in \mathcal{F}_{t-\delta t}$ one has $\mathbb{E}(X \mathbb{I}_A) = \mathbb{E}(Y \mathbb{I}_A)$, and as Ω is finite, it suffice to check this property in the case A is an atom of $\mathcal{F}^S_{t-\delta t}$, i.e.

$$A = \{S_{\delta t} = s_{\delta t}, \ldots, S_{t-\delta t} = s_{t-\delta t}\} \; ;$$

but in that case we have

$$\begin{aligned}
\mathbb{E}(X\mathbb{I}_A) &= \mathbb{E}(f(S_{\delta t}, \ldots, S_{t-\delta t}, S_t) \mathbb{I}_{\{S_{\delta t}=s_{\delta t}, \ldots, S_{t-\delta t}=s_{t-\delta t}\}}) \\
&= \mathbb{E}(f(s_{\delta t}, \ldots, s_{t-\delta t}, s_{t-\delta t} U_t) \mathbb{I}_{\{S_{\delta t}=s_{\delta t}, \ldots, S_{t-\delta t}=s_{t-\delta t}\}}) \\
&= \mathbb{E}(f(s_{\delta t}, \ldots, s_{t-\delta t}, s_{t-\delta t} U_t)) \mathbb{E}(\mathbb{I}_{\{S_{\delta t}=s_{\delta t}, \ldots, S_{t-\delta t}=s_{t-\delta t}\}}) \text{ by independence,} \\
&= (pf(s_{\delta t}, \ldots, s_{t-\delta t}, s_{t-\delta t} u) + (1-p) f(s_{\delta t}, \ldots, s_{t-\delta t}, s_{t-\delta t} d)) \mathbb{E}(\mathbb{I}_A) \\
&\qquad\qquad \text{as } U_t := u^{\delta J_t} d^{1-\delta J_t}, \text{ with } \delta J_t \rightsquigarrow \mathcal{B}(1,p), \\
&= \mathbb{E}((pf(s_{\delta t}, \ldots, s_{t-\delta t}, s_{t-\delta t} u) + (1-p) f(s_{\delta t}, \ldots, s_{t-\delta t}, s_{t-\delta t} d)) \mathbb{I}_A) \text{ by linearity of } \mathbb{E}, \\
&= \mathbb{E}((pf(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} u) + (1-p) f(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} d)) \mathbb{I}_A) \\
&= \mathbb{E}(Y \mathbb{I}_A).
\end{aligned}$$

□

Applying lemma 3.5 and using the fact that S and M are martingales, we have

$$S_{t-\delta t} = \mathbb{E}(S_t \mid \mathcal{F}^S_{t-\delta t}) = pS_{t-\delta t} u + (1-p) S_{t-\delta t} d =: pS^+_{t-\delta t} + S^-_{t-\delta t},$$

and, as $M_t \in \mathcal{F}^S_t$, it exists $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ such that $M_t = f(S_{\delta t}, \ldots, S_t)$, thus

$$\begin{aligned}
M_{t-\delta t} = \mathbb{E}(M_t \mid \mathcal{F}^S_{t-\delta t}) &= pf(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} u) + (1-p) f(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} d) \\
&=: pM^+_{t-\delta t} + (1-p) M^-_{t-\delta t}.
\end{aligned}$$

From these two identities we deduce trivially that

$$\begin{aligned}
p(M^+_{t-\delta t} - M_{t-\delta t}) &= (p-1)(M^-_{t-\delta t} - M_{t-\delta t}) \text{ , and} \\
p(S^+_{t-\delta t} - S_{t-\delta t}) &= (p-1)(S^-_{t-\delta t} - S_{t-\delta t}) \text{ , and thus} \\
\frac{M^+_{t-\delta t} - M_{t-\delta t}}{S^+_{t-\delta t} - S_{t-\delta t}} &= \frac{M^-_{t-\delta t} - M_{t-\delta t}}{S^-_{t-\delta t} - S_{t-\delta t}} =: \alpha.
\end{aligned} \tag{3.9}$$

We see that, defined in this way, $\alpha \in \mathcal{F}^S_{t-\delta t}$, and, as $S_t = S_{t-\delta t} U_t$, with $U_t \in \{u, d\}$, we deduce from (3.9) that for any of these two values of U_t we have

$$\alpha \delta S_{k\delta t} = \alpha(S_{t-\delta t} U_t - S_{t-\delta t}) = f(S_{\delta t}, \ldots, S_{t-\delta t}, S_{t-\delta t} U_t) - M_{t-\delta t} = \delta M_t,$$

so it suffices to choose $\alpha_{k\delta t} := \alpha$, which is $\mathcal{F}^S_{t-\delta t}$-measurable. □

Remark: Hedge-ratio : Observe that (3.9) leads to

$$\alpha_{k\delta t} = \alpha = \frac{M^+_{t-\delta t} - M^-_{t-\delta t}}{S^+_{t-\delta t} - S^-_{t-\delta t}} \tag{3.10}$$

which, when $M_t = \mathbb{E}(X \mid \mathcal{F}^S_t)$ is the value of the hedge-portfolio of some european option with $\mathcal{F}^S_T$-measurable pay-off X, is called the *hedge-ratio* of the option, and is the ratio between $M^+_{t-\delta t} - M^-_{t-\delta t}$, the difference of the two possible next-coming values of the option, and $S^+_{t-\delta t} - S^-_{t-\delta t}$, the difference of the two possible next-coming values of the stock. So, this hedge-ratio Δ is the (preditable) quantity of stock that the hedge-portfolio should include at each time.

3.4 Appendix 1 : self-financing strategies

(Here should come some words on the so called self-financing strategies.)

3.5 Appendix 2 : hedging away the risk of options : the Black and Scholes Story

Trading options on future prices goes probably back as far as the uses of money for exanging goods. Trading an option is exchanging the risk related to the fluctuations of the price of products like rice, coffee, or gold. Risk management is a work for itself : a farmer or a manufacturer should not keep such a risk : he should get an insurance at the best possible price and "externalize" the risk to somebody whose business is risk. The largest place were such "derivatives" are traded is the CBOT, the Chicago Board Of Trades. At the begining of the 1970', F. Black and M. Scholes invented a technique similar to the CRR technique we have exposed here (and that was invented later, after an idea of Nobel Prize winner Scharp), that would give the CRR-result after passing to the limit on large n. Their modelisation was not based on stochastic ideas but inspired by the diffusion of heat and numerical analysis of the heat equation, using Fourier transforms. They also had an extra clever idea : put the numerical algorithms in an hand-held computer. They sold this machine to option traders who discovered the miracle of being able to hedge away (most of) the risks related to their job. The CBOT understood that this would boost its activities and asked Black and Scholes to disclose their technique, what they did. R.C. Merton, at the same time, published independently equivalent results, using stochastic techniques, the modeling requiring perhaps less financial/physical knoledge.

The estime of option traders for the findings of Black and Scholes that it has been reported that once B&S visited the NYSE, the New-York Stock Exchange, when they appear on the visitors galery, some trader noticed they where there, turned towards them, slapped his hands and this turned into an (obviously standing...) ovation for the people that helped them so efficently to do their job.

Black died in the 80'. Scholes and Merton received the Nobel Prize in 1997, of course not forgetting that Black was, together with them, at the origin of the story. They also mentioned that actually the ideas they used could also have been found in thesis defended in Paris in 1900 by Bachelier, but this is an other story.

Exercise 3.1 *Consider a process $M = (M_t)_{t\in[0..T]_{\delta t}}$ that is not stochastic (i.e. $M_t = f(t)$ for some function f). How should f be chosen in order that M be a martingale ?*

Exercise 3.2 *For the two-time-steps case of exercise 1.1, denote by $\alpha = (\alpha_t)_{t\in\{\delta t, 2\delta t\}}$ the hedging strategy and let x be its* premium *(i.e. the value x such that $x + P\&L^S_T(\alpha) = (K - S_T)^+$. We consider now the result of the buy-and-hold strategy $\alpha' = (\alpha'_t)_{t\in\{\delta t, 2\delta t\}}$ that is constant, with $\alpha'_t = \alpha_{\delta t}$ for any t (i.e. you forget to rebalance your portfolio at time δt). Determiner the various values of the r.v. $x + P\&L^S_T(\alpha') - (K - S_T)^+$ (the overall profit and loss you made by neglecting to rebalance). You may wish to define $\Omega := \{u, d\}^2$.*

Chapter 4

Arbitrage probabilities

4.1 Uncomplete markets

The CRR model we considered up to here has a somewhat ideal feature : it allows to hedge exactly (or "duplicate") any contingent claim (derivative) written on the stock (-model). Actually, the other popular model for a stock, namely the Black-Scholes model, has also this property[1]. If you find that this property is a bit too idealistic then you will be interested in the so-called problem of "uncomplete markets". For the mathematical financist, an *uncomplete market* is a stock model S for which some $\mathcal{F}_T^S$-measurable contingent claim with payoff Π_T can not be hedged, i.e. there is no $(\mathcal{F}_t^S)_{t\leq T}$-predictable process $\alpha = (\alpha_t)_{t\leq T}$ such that $P\&L_T^S(\alpha) = \Pi_T - \Pi_0$ for some (nonrandom) premium Π_0. In such a model the (up to here implicit) method of pricing an option by the value of its hedging portfolio can no longer be applied and has to be generalised into the so-called arbitrage method : it consists in rejecting any model leading to (model) strategy for which you could, riskless, make more money then investing the same amount on an account paying the fixed interest rate. Indeed, if such an *arbitrage* strategy would exist, it would leed to large volumes of transactions all in the same direction (buy or sell) that would make the model irrealistic. Actually, some arbitrage do exist sometimes, but only for a short time (about 30") as banks do hire "arbitragers" to take advantage of it to the benefit of the bank. So arbitrage modelling can be considered as a negative approach : it only allows you to reject models ; but, as arbitrage-free uncomplete models do exist, it turns out to be a very usefull modelling tool when you are interested in improoving the existing models.

Before we introduce a more general setting for stock models let's give the most elementary example of stock that is not complete : we take it from the beautiful introduction to mathematical finance of Stanley Pliska [2], who, by the way is one of the founder of modern finance ; there is only one time step $\delta t = T$, $S_0 = 5$, and $S_{\delta t}$ takes three (and not only two) values 3, 4, and 6. Consider a derivative on S, with payoff $\pi(S_{\delta t})$. If we want to hedge it with α stocks and a β riskless investment, (α, β) should satisfy

$$\begin{cases} \alpha 3 + \beta R & = \pi(3) \\ \alpha 4 + \beta R & = \pi(4) \\ \alpha 6 + \beta R & = \pi(6) \text{ with } R := e^{r\delta t} = 1, \text{ as we assume } r = 0, \end{cases}$$

which is a system of three equations and only two unknown, so has no solution (unless $\pi(6) - 3\pi(4) + 2\pi(3) = 0$). This example suggests how the completeness of the CRR model is related with the fact that the law of $\delta S_{t+\delta t}$ knowing $\mathcal{F}_t$ is a binary law.

[1] provided you accept the idea to hedge (buy and sell the stock for your hedging portfolio) as you would ajust the stearing wheel of your car when driving and observing how the available space in front of you evolves, however in a somewhat more shaking way, as your command will be only $\mathcal{C}^0$.

4.2 Profit-and-Loss and martingales

In order to stay radically elementary we go on assuming that Ω is finite, $\mathbb{P}(\{\omega\}) > 0$ for any $\omega \in \Omega$, and that there is a finite number of time steps $t \in \mathbb{T} := [0..T]_{\delta t}$, $n\delta t = T$; let S be any *stochastic process* $S : \Omega \times \mathbb{T} \longrightarrow \mathbb{R}$, $S(\omega, t) =: S_t(\omega)$. We model the *information available at time*[2] t by an algebra $\mathcal{F}_t \subseteq \mathcal{P}(\Omega)$, such that $\mathbb{F} := (\mathcal{F}_t)_{t\in[0..T]_{\delta t}}$ is a filtration (information does not get lost), and such that each r.v. S_t is $\mathcal{F}_t$-measurable (the price of the stock at time t belongs to the information available at time t) ; so $\mathcal{F}_t^S \subseteq \mathcal{F}_t$ but is not necessarily equal. A *market* (model) is such a triple $(\Omega, S, \mathbb{F})$.

In this more general setting we define a *predictable strategy* as an $\mathbb{F}$-predictable process $\alpha = (\alpha_t)_{t\in[0..T]_{\delta t}}$ (i.e. $\alpha_t \in \mathcal{F}_{t-\delta t}$ for any $t \in]0..T]_{\delta t}$), and its Profit and Loss on S is again

$$P\&L_t^S(\alpha) = \sum_{s\in]0..T]_{\delta t}} \alpha_s \delta S_s \text{ , where } \delta S_s := S_s - S_{s-\delta t}.$$

Proposition 4.1 *Let $(\Omega, S, \mathbb{F})$ be any market, and let $\mathbb{P}^*$ be a probability on Ω ; denote by $\mathbb{E}^*$ the expectation with respect to $\mathbb{P}^*$. Then S is a $(\mathbb{P}^*, \mathbb{F})$-martingale if and only if $\mathbb{E}^*(P\&L_T(\alpha)) = 0$ for any $\mathbb{F}$-predictable strategy.*

Proof: To show that S is a martingale we shall apply proposition 3.1. As $\mathcal{F}_t^S \subseteq \mathcal{F}_t$ for any t, S is $\mathbb{F}$-adapted. Now, for any $t_0 \in [0..T[_{\delta t}$ and any $A \in \mathcal{F}_{t_0}$, define $\alpha_t = \alpha_t^{A,t_0}$

$$\alpha_t^{A,t_0} := \begin{cases} \mathbb{I}_A & \text{if } t = t_0 + \delta t \geq 0 \\ 0 & \text{if } t \neq t_0 + \delta t \geq 0 \end{cases}$$

which means that if A is true at $t = t_0$, then buy one stock and sell it immediately after, at $t = t_0 + \delta t$. So α_t is deterministic if $t \neq t_0 + \delta t$, and $\mathcal{F}_{t_0}$-measurable if $t = t_0 + \delta t$, thus $\alpha = (\alpha_t)_{t\in[0..T]_{\delta t}}$ is $\mathbb{F}$-predictable. Now

$$P\&L_T^S(\alpha) = \sum_{t\in]0..T]_{\delta t}} \alpha_t \delta S_t = \mathbb{I}_A \delta S_{t_0+\delta t},$$

thus, by assumption,

$$0 = \mathbb{E}^*(P\&L_T^S(\alpha)) = \mathbb{E}^*(\mathbb{I}_A \delta S_{t_0+\delta t}),$$

and this is true for any $t_0 \in [0..T[_{\delta t}$ and any $A \in \mathcal{F}_{t_0}$, so $\mathbb{E}^*(\delta S_{t_0+\delta t} \mid \mathcal{F}_{t_0}) = 0$ for any $t_0 \in [0..T[_{\delta t}$. So, by proposition 3.1, S as a martingale.

Conversely, if S is a $(\mathbb{P}^*, \mathbb{F})$-martingale, then again by proposition 3.1, $\mathbb{E}^*(\delta S_s \mid \mathcal{F}_{s-\delta t}) = 0$ for any $s \in]0..T]_{\delta t}$; let α be any $\mathbb{F}$-predictable strategy ; so α_s is $\mathcal{F}_{s-\delta t}$-measurable ; we will thus condition on $\mathcal{F}_{s-\delta t}$ and apply the conditioning trick (proposition 0.6, 5) :

$$\begin{aligned} \mathbb{E}^*(P\&L_T^S(\alpha)) = \mathbb{E}^*\left(\sum_{s\in]0..T]_{\delta t}} \alpha_s \delta S_s\right) &= \sum_{s\in]0..T]_{\delta t}} \mathbb{E}^*(\mathbb{E}^*(\alpha_s \delta S_s \mid \mathcal{F}_{s-\delta t})) \\ &= \sum_{s\in]0..T]_{\delta t}} \mathbb{E}^*(\alpha_s \mathbb{E}^*(\delta S_s \mid \mathcal{F}_{s-\delta t})) = 0. \end{aligned}$$

□

[2]in the sens that a r.v. Y is $\mathcal{F}_t$-measurable if and only if it can be considered to be no longer unknown at time t, beeing a deterministic function of random variables like $\mathbb{I}_{A_1}, \ldots, \mathbb{I}_{A_m}$, $A_i \in \mathcal{F}_t$, "already observed at time t", and that caracterize the information $\mathcal{F}_t$.

4.3 Arbitrage-free markets

4.3.1 The fundamental theorem of arbitrage-free probabilities

Definition: We say that an $\mathbb{F}$-predictable strategy α is an *arbitrage strategy* (or simply an *arbitrage*) of the market $(\Omega, S, \mathbb{F})$ if and only if $P\&L_T^S(\alpha) \geq 0$, and it exists $\omega_0 \in \Omega$ such that $P\&L_T^S(\alpha)(\omega_0) > 0$.

In other words, an arbitrage strategy is a predictable strategy for which you begin with no money, borrow the amount $\alpha_{\delta t} S_0$ (and possibly more or less after, in a self-financing strategy), you end up with no debt in any state of the world $\omega \in \Omega$ ($P\&L_T^S(\alpha)(\omega) \geq 0$) and you make money at least in one state $\omega_0 \in \Omega$. As already mentioned, such a model with arbitrage does not seem realistic so one is only interested in *arbitrage-free* markets $(\Omega, S, \mathbb{F})$, i.e. that admit no arbitrage.

Observe that the definiton of arbitrage involves no probability. The next theorem will show how a probability can help to express absence of arbitrage, that will turn out to be also an efficient pricing tool.

Theorem 4.2 *A (finite) market $(\Omega, S, \mathbb{F})$ is arbitrage free if and only if it exists a probability $\mathbb{P}^*$, $\mathbb{P}^*(\{\omega\}) > 0$ for any $\omega \in \Omega$, such that $S = (S_t)_{t \in [0..T]_{\delta t}}$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale.*

Observe that, as we require $\mathbb{P}(\{\omega\}) > 0$ for any $\omega \in \Omega$, if α is an arbitrage strategy, necessarly $\mathbb{E}(P\&L_T(\alpha)) > 0$. Before proving the theorem, let us first show examples and partial results that will help in the proof. Sufficency will be shown at corollary 4.4, and next section will be devoted to the the existence of $\mathbb{P}^*$.

Example: Let us come back on Pliska's example ; recall $\delta t = T$, $S_0 = 5$, and $S_{\delta t} \in \{3, 4, 6\}$. Define $p = \mathbb{P}^*\{S_{\delta t} = 3\}$, $q = \mathbb{P}^*\{S_{\delta t} = 4\}$, and $1 - p - q = \mathbb{P}^*\{S_{\delta t} = 6\}$. Assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$; $S = (S_t)_{t \in \{0,T\}}$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale if and only if

$$5 = S_0 = \mathbb{E}^*(S_{\delta t} \mid \mathcal{F}_0) = \mathbb{E}^*(S_{\delta t}) = p3 + q4 + (1 - p - q)6 = 6 - 3p - 2q,$$

or, equivalently, $q = \frac{1}{2} - \frac{3}{2}p$ (and $1 - p - q = \frac{1}{2} - \frac{1}{2}p$). So, finally, S is a martingale if and only if $3p + 2q = 1$ and $\mathbb{P}^*\{S_{\delta t} = 3\} = p \in (0, \frac{1}{3})$; this leads to $q = \mathbb{P}^*\{S_{\delta t} = 4\} \in (0, \frac{1}{2})$, and $1 - p - q = \mathbb{P}^*\{S_{\delta t} = 6\} \in (\frac{1}{2}, \frac{2}{3})$. So, according to theorem 4.2, Pliska's model is arbitrage-free if and only if $p \in (0, \frac{1}{3}) =: (p^{*-}, p^{*+})$.

Exercise: Assume $\emptyset \neq \{S_{\delta t} > 5\} \in \mathcal{F}_0$; give an example of an arbitrage strategy ; check that there is no probability $\mathbb{P}^*$ such that S is a $(\mathbb{P}^*, \mathbb{F})$-martingale. Same questions when assuming that $\{S_{\delta t} = 4\} \in \mathcal{F}_0$.

Proposition 4.3 *Assume S is a $(\mathbb{P}^*, \mathbb{F})$-martingale and α is any $\mathbb{F}$-predictable strategy. Then $P\&L(\alpha)$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale.*

Proof: In proof of proposition 3.3, simply replace $\mathbb{F}(S)$ by $\mathbb{F}$. □

Corollary 4.4 *If it exists a probability $\mathbb{P}^*$ such that S is a $(\mathbb{P}^*, \mathbb{F})$-martingale, then the finite market $(\Omega, S, \mathbb{F})$ is arbitrage free.*

Proof: Let $\alpha = (\alpha_t)$ be any $\mathbb{F}$-predictable strategy. By proposition 4.3 $P\&L(\alpha)$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale. So $P\&L_0(\alpha) = \mathbb{E}^*(P\&L_T(\alpha) \mid \mathcal{F}_0)$; but by definition $P\&L_0(\alpha) = 0$, thus

$$\mathbb{E}^*(P\&L_T(\alpha)) = \mathbb{E}^*(\mathbb{E}^*(P\&L_T(\alpha) \mid \mathcal{F}_0)) = \mathbb{E}^*(P\&L_0(\alpha)) = \mathbb{E}^*(0) = 0.$$

So α is not an arbitrage. □

4.3.2 Existence of $\mathbb{P}^*$

Let us now proof the reciprocal of corollary 4.4 and thus finish the proof of theorem 4.2. So we assume that the finite market $(\Omega, S, \mathbb{F})$ is arbitrage free, and we want to construct a probability $\mathbb{P}^*$ for which the assumption of proposition 4.1 hold, namely the expectation $\mathbb{E}^*(P\&L_T(\alpha))$ of the final profit and loss $P\&L_T(\alpha)$ of any predictable strategy α is 0. Recall the *separation theorem* (also called Hahn-Banach theorem in the case of an infinite dimensional Banach space) : *Let $H \subseteq \mathbb{R}^d$ be any proper subspace and $\Gamma \subseteq \mathbb{R}^d$ be any compact, convex subset. If $H \cap \Gamma = \emptyset$, then it exists a linear map $\Lambda : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $\Lambda(h) = 0$ for any $h \in H$, and $\Lambda(\gamma) > 0$ for any $\gamma \in \Gamma$.* Let us apply the separation theorem in the following case. $\mathbb{R}^d = \mathbb{R}^\Omega$, the finite dimensional space of all r.v. on Ω, in which we consider the two following subsets :

$$\begin{aligned} H &:= \{P\&L_T(\alpha) \mid \alpha \ \mathbb{F}\text{-predictable}\}, \text{ and} \\ \Gamma &:= \{X \in (\mathbb{R}^+)^\Omega \mid \sum_{\omega\in\Omega} X(\omega) = 1\}. \end{aligned}$$

Clearly H and Γ satify the conditions of the separation theorem ; in particular, the fact that the market $(\Omega, S, \mathbb{F})$ is arbitrage free implies that $H \cap \Gamma = \emptyset$. Let $\Lambda : \mathbb{R}^\Omega \to \mathbb{R}$ be the corresponding separation linear map ; define the λ_ω, $\omega \in \Omega$, be such, that for any $X \in \mathbb{R}^\Omega$, $\Lambda(X) = \sum_{\omega\in\Omega} \lambda_\omega X(\omega)$ (coordinates of Λ in the bases of the space of linear forms on $\mathbb{R}^\Omega$ composed of the projections $X \mapsto X(\omega)$, $\omega \in \Omega$). Let's check that it suffices to define $\mathbb{P}^*$ by $\mathbb{P}^*(\omega) = p_\omega = \frac{1}{|\Lambda|}\lambda_\omega$, where $|\Lambda| := \sum_{\omega\in\Omega} \lambda_\omega$. Indeed, considering for any $\omega_0 \in \Omega$ the r.v. $\mathbb{I}_{\{\omega_0\}} \in \Gamma$ shows that $0 < \Lambda(\mathbb{I}_{\{\omega_0\}}) = \lambda_{\omega_0}$, so $p_\omega > 0$ for all $\omega \in \Omega$, and obviously $\sum_{\omega\in\Omega} p_\omega = 1$. So this defines a probability $\mathbb{P}^*$ such that $\mathbb{P}^*(\{\omega\}) > 0$ for any $\omega \in \Omega$, and we have

$$\mathbb{E}^*(X) = \sum_{\omega\in\Omega} p^*_\omega X(\omega) = \frac{1}{|\Lambda|}\Lambda(X),$$

so $\mathbb{E}^*(P\&L_T(\alpha)) = 0$ for any predictable strategy α, as $P\&L_T(\alpha) \in H \subseteq \text{Ker }(\Lambda)$. Now, by proposition 4.1, this implies that S is a $(\mathbb{P}^*, \mathbb{F})$-martingale.

□

4.3.3 Pricing with an arbitrage-free probability

Definition: A number $x \in \mathbb{R}$ is called an *overhedging price* for the $\mathcal{F}^S_T$-measurable r.v. X (possibly the pay-off of some option on S) if and only if it exists a predictable strategy α on S such that

$$x + P\&L_T(\alpha) \geq X \ ; \tag{4.1}$$

x is an *underheging price* for X if it exists a predictable strategy α on S such that

$$x + P\&L_T(\alpha) \leq X. \tag{4.2}$$

Proposition 4.5 *Assume the model S is arbitrage-free, and let $\mathbb{P}^*$ be any probability on Ω such that S is a $(\mathbb{P}^*, \mathbb{F}(S))$-martingale. Let x^+ be any overhedging price and x^- be any underhedging price for some $\mathcal{F}^S_T$-measurable r.v.X ; then*

$$x^- \leq \mathbb{E}^*(X) \leq x^+.$$

Proof: By definition, it exists a predictable overhedging strategy α such that $x^+ + P\&L_T^S(\alpha) \geq X$. By proposition 4.1, $\mathbb{E}^*(P\&L_T^S(\alpha)) = 0$. Thus

$$\mathbb{E}^*(X) \leq \mathbb{E}^*(x^+ + P\&L_T^S(\alpha)) \leq x^+ + \mathbb{E}^*(P\&L_T^S(\alpha)) = x^+.$$

One would show that $x^- \leq \mathbb{E}^*(X)$ in a similar way, using a predictable underhedging strategy. $\square$

At theorem 4.2 we have seen that for any arbitrage-free stock S it exists at least one martingale probability $\mathbb{P}^*$ for S. So $x^* := \mathbb{E}^*(X)$ bounds from below all the overhedging prices and from above all the underhedging prices. Define

$$\begin{aligned} x_X^+ &:= \text{Min}\,\{x \in \mathbb{R}, \text{ such that } x \text{ is an overhedging price for } X\}, \text{ and similarly} \\ x_X^- &:= \text{Max}\,\{x \in \mathbb{R}, \text{ such that } x \text{ is an underhedging price for } X\}. \end{aligned}$$

We just have shown that $x^* \in [x_X^-, x_X^+] \neq \emptyset$; this interval is called the interval of arbitrage-free prices. Observe that, as Ω and $[0..T]_{\delta t}$ are finite, the Max and the Min are achieved, so it exists predictable strategies α^+ and α^- such that $x_X^- + P\&L_T^S(\alpha^-) \leq X \leq x_X^+ + P\&L_T^S(\alpha^+)$, and it exist ω^- and ω^+ (possibly equal) such that $x_X^\pm + P\&L_T^S(\alpha^\pm)(\omega^\pm) = X(\omega^\pm)$.

Exercise 4.1 In Pliska's example of section 4.1, we defined $(p^{*-}, p^{*+}) := (\frac{1}{2}, \frac{2}{3})$. Check that for the at-the-money Call $\varphi(S_T) := (S_T - S_0)^+$, one has $x^+ = \mathbb{E}^+(\varphi(S_T))$ and $x^- = \mathbb{E}^-(\varphi(S_T))$, where $\mathbb{E}^+$ stands for the expectation with respect to the probability defined using $p = p^{*+}$ and similar for $\mathbb{E}^-$. Compute the overall profit-and-losses $P\&L_T^S(\alpha^+) - \varphi(S_T) + x^+$ and $P\&L_T^S(\alpha^-) - \varphi(S_T) + x^-$.

Proposition 4.6 *Assume it exists a probability $\mathbb{P}^*$ such that S is a $(\mathbb{P}^*, \mathbb{F}(S))$-martingale that has the martingale representation property. Then $x^- = x^* = x^+$, the market is arbitrage-free and is complete.*

Proof: Let $M_t := \mathbb{E}^*(X \mid \mathcal{F}_t^S)$. By proposition 3.2, $M := M(T, X) = (\mathbb{E}^*(X \mid \mathcal{F}_t^S))_{t \in [0..T]_{\delta t}}$ is a martingale, and has thus, by hypothesis, a representation α, $M_t = M_0 + P\&L_t^S(\alpha) = x^* + P\&L_t^S(\alpha)$. Now, as X is $\mathcal{F}_T^S$-measurable, $X = \mathbb{E}^*(X \mid \mathcal{F}_T^S) = M_T = x^* + P\&L_T^S(\alpha)$, so α is both an over and underhedging strategy, and $x^- = x^* = x^+$. $\square$

Remark: *Selling* an option at any price $x > x^+$ allows to make an arbitrage. Indeed : get the premium (price) x, keep it, and apply the strategy α^+. At time T you make a profit-and-loss $P\&L_T^S(\alpha^+)(\omega) \geq X(\omega) - x^+$, where ω is the state of the world it turned out you were living in. You pay $X(\omega)$ and kept x, so you have

$$x + P\&L_T^S(\alpha^+)(\omega) - X(\omega) \geq x - x^+.$$

So you have at least $x - x^+ > 0$ for a "free-lunch".

Exercise 4.2 Show how to get a free-lunch for *buying* an option with pay-off X at a price $x < x^-$.

Exercise 4.3 We hope that, when applying the previous remark or exercise 4.2 you did not forget that we assumed that $r = 0$. If, most probably, the overnight interest is positive ($r > 0$), you will have to apply the remark in section 3.0. So rewrite all the above results for $\tilde{S} = (e^{-rt} S_t)_{t \in [0..T]_{\delta t}}$, with $\tilde{\alpha}$, $\tilde{x}^\pm$, $\tilde{\alpha}^\pm$, and find the correct value of x_X^- and x_X^+ for the interval of arbitrage-free prices $[x_X^-, x_X^+]$.

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Chapter 5

American Options

Whereas a european option gives to its holder the right (and get his pay-off) at a fixed time T, the corresponding american option gives this right at *any* time $t \in (0..T]_{\delta t} := \{\delta t, 2\delta t, \dots, T = n\delta t\}$ between 0 et T. For instance, a european call on some asset S_t will give $(S_T - K)^+$ at time T and the american call on the same underlying asset will, if exercised on time $t \leq T$, give the pay-off $\varphi(S_t) = (S_t - K)^+$. In this chapter, we will describe how to compute the price of an american option and, by the way, we will meet some nice tools from stochastic calculus such as the theorem of optimal stopping time or the Doob-Meyer decomposition of supermartingales.

5.1 Backward-induction computing of the price

Just as before, the process (S_t), defined for all $t \in [0..T]_{\delta t} := \{0, \delta t, 2\delta t, \dots, T = N\delta t\}$, models the evolution of a financial asset as time goes, and we assume that this process is adapted to the filtration $\mathcal{F} = (\mathcal{F}_t)$, that models the available information at time t. Let U_t be the value at time t of an american option with pay-off $\varphi(S_t)$: if he exercices his option at time t, the holder gets $\varphi(S_t)$). How to compute the price of this option ? Just as for the european option we will compute by (backward) induction, beginning with final value, the minimal value of a hedging portfolio. First, if the option has not been exercised by this time, the value of the hedging portfolio at the final time T has to be the value of the pay-off $\varphi(S_T)$, just like for a european option. At the preceding instant, $t = T - \delta t$, the seller will need to have a wealth at least equal to the pay-off $\varphi(S_{T-\delta t})$, in case the holder of the option would exercise at this time, and at the same time at least equal to $e^{-r\delta t}\mathbb{E}(\varphi(S_T)/\mathcal{F}_{T-\delta t})$ which is the price of the hedging portfolio that will allow him to face his commitments the next time, T, in case the holder does not exercice on the present time. So the price of the american option at $T - \delta t$ is :

$$U_{T-\delta t} = \text{Max}\,\{\varphi(S_{T-\delta t}), e^{-r\delta t}\mathbb{E}(\varphi(S_T)/\mathcal{F}_{T-\delta t})\} = \text{Max}\,\{\varphi(S_{T-\delta t}), e^{-r\delta t}\mathbb{E}(U_T/\mathcal{F}_{T-\delta t})\}.$$

Of course, this reasoning is also valid at $t = T - 2\delta t$, $t = T - 3\delta t$, and so on. In this way, we find the following backward induction formula :

$$\begin{cases} U_t &= \text{Max}\,\Big(\varphi(S_t), e^{-r\delta t}\mathbb{E}(U_{t+\delta t}/\mathcal{F}_t)\Big) \\ U_T &= P_T \end{cases} \tag{5.1}$$

In case of european option, as for instance for a Call option, the backward induction formula could be written $C_t = e^{-r\delta t}\mathbb{E}(C_{t+\delta t}/\mathcal{F}_t)$ and thus get the fundamental formula $C_t = e^{-r(T-t)}\mathbb{E}(C_T/\mathcal{F}_t)$ that shows that the value at time t is the discounted expectation of its pay-off (for the risk-neutral probability). In the case of an american option it is not possible to derive from this relation (5.1) the value of U_t directly as a function of t and of the pay-off, but we will see that it exist a kind of similar “closed formula”, but less explicit. On the other hand, it is easy to program this induction formula in order to compute the premium U_t at any time t.

It should not be a surprise that an american option is more expansive as its european counterpart, or at least just as expansive, as it gives access to more rights (in the choice of the exercise time $t \leq T$.) The difference between both is called the *early exercice premium.* In what case would it be the holder's interest to use his early exercise right, i.e. in what situations is this premium positive ? We will see that it is not the case for a Call option, unless the underlying asset would pay a dividend in the meantime, but it can be the case for a Put option, unless we assume that the interest-rate r vanishes, which is not very realistic.

Proposition 5.1 *The underlying asset S_t being as above (no dividen paying), the price of an american Call option on S_t is equal to the price of a european Call option with same exercise time T and same exercise price K.*

Proof: Formula (5.1) implies $U_{t+\delta t} \geq \varphi(S_{t+\delta t})$ for all t. As conditional expectation and discounting keeps this inequality, we have

$$e^{-r\delta t}\mathbb{E}(U_{t+\delta t}/\mathcal{F}_t) \geq e^{-r\delta t}\mathbb{E}(\varphi(S_{t+\delta t})/\mathcal{F}_t).$$

As $\varphi(S_t) = (S_t - K)^+$ is a convex Jensen's inequality[1]implies

$$e^{-r\delta t}\mathbb{E}(U_{t+\delta t}/\mathcal{F}_t) \geq \left(e^{-r\delta t}\mathbb{E}(S_{t+\delta t}/\mathcal{F}_t) - e^{-r\delta t}K\right)^+.$$

But, as the discounted value of S_t is a martingale, $e^{-r\delta t}\mathbb{E}(S_{t+\delta t}/\mathcal{F}_t) = S_t$, and thus

$$e^{-r\delta t}\mathbb{E}(U_{t+\delta t}/\mathcal{F}_t) \geq \left(S_t - e^{-r\delta t}K\right)^+ \geq (S_t - K)^+ = \varphi(S_t),$$

The last inequality simply follows from the fact that, as $r \geq 0$, $-e^{-r\delta t} \geq -1$. So, in the two terms in the maximum of (5.1), the second one stays, for all t, larger or equal to the first one. So (5.1) reduces to the same formula as for a european Call option which implies that both european and american options are equal. □

Observe that, when replacing the pay-off function of a Call option $(S_t - K)^+$ by the one of a Put option $(K - S_t)^+$, the last inequality is no-longer valid as soon as $r > 0$. *De facto*, if early exercise is never more profitable in the case of a Call option, it is often the the case in the case of a Put option (unless $r = 0$), as we shall see now.

5.2 The optimal stopping time theorem

It is difficult to see when, in the induction formula (5.1) the maximum in this formula will be equal to the immediate pay-off $\varphi(S_t)$ and early exercice would be optimal for the option holder. Actually, it exists a curve in the (t, S_t) space, called *exercise line* (see figure 5.1), with following property : as long as the price of the underlying asset S_t does not cross this line, early exercise value is less than the value of the option and it is preferable to keep the option. But as soon as the price S_t crosses the line, the option holder should exercise as immediate pay-off is larger as the value of the option. There is no explicit formula for this curve but it can be computed numericaly. From a theoretical point of view it can be shown that this exercise line is the set of (τ, S_τ) for some $\mathcal{F}$-stopping time called *optimal stopping-time.* One has the following theorem :

Theorem 5.2 *Let $\mathcal{T}(t,T)$ be the set of all $\mathcal{F}$-stopping-times with value in $[t..T]_{\delta t}$. The price at time t of the american option with pay-off function $\varphi(S_t)$ is given by*

$$U_t = \max_{\tau \in \mathcal{T}(t,T)} e^{-r(T-t)}\mathbb{E}(\varphi(S_\tau)/\mathcal{F}_t) = e^{-r(T-t)}\mathbb{E}(\varphi(S_{\tau_t})/\mathcal{F}_t)$$

the maximum being achieved for the stopping-time τ_t defined by

$$\tau_t := Min\{s \in [t..T]_{\delta t} \ \ , \ \ U_s = \varphi(S_s)\}.$$

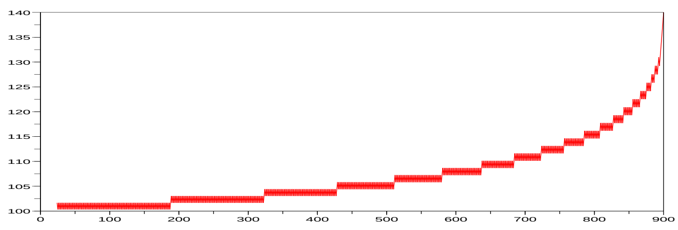

Figure 5.1: Plotting of the exercise line of an american at-the-money Put in a Cox, Ross, Rubinstein model, with $S_0 = 140$, $\sigma = 0.4$, $r = 0.05$, $T = 1$, and $n = 900$.

In particular, if we apply this theorem for $t = 0$, the premium U_0 of an american option is equal to $U_0 = e^{-rT}\mathbb{E}(\varphi(S_{\tau_0}))$, where τ_0 is the first time where the price of the option will be equal to the pay-off, this is the first time where the maximum in formula (5.1) is equal to the first term. More precisely, as long as this maximum is equal to the second term (the expectation of future values) it is not necessary to exercise, but as soon as the immediate pay-off $\varphi(S_t)$ is larger than the hedging of the future, the holder should exercise his right (as he could get the option for a lower price).

Proof: Without loss of generality we can assume that $t = 0$.

Let us consider the discounted value $\tilde{U}_t$ of the american option that is defined by $\tilde{U}_t = e^{-rt}U_t$, and let us consider $\tilde{U}_{t\wedge\tau_0}(\omega)$ that is equal to $\tilde{U}_t(\omega)$ as long as $t < \tau_0(\omega)$ and that is constant $\tilde{U}_{\tau_0(\omega)}(\omega)$ for all $t \geq \tau_0(\omega)$. This walk is called the walk $\tilde{U}$ stopped at τ_0. We shall check that this walk is an $\mathcal{F}$-martingale : by definition of τ_0, as $1 = \mathbb{I}_{t<\tau_0} + \mathbb{I}_{t\geq\tau_0}$ and as these two charactéristic functions are $\mathcal{F}_t$-mesurable, as τ_0 is a $\mathcal{F}$-stopping time, we have

$$\begin{aligned}
\mathbb{E}(\tilde{U}_{t\wedge\tau_0} - \tilde{U}_{(t+\delta t)\wedge\tau_0}/\mathcal{F}_t) &= \mathbb{I}_{t<\tau_0}\mathbb{E}(\tilde{U}_{t\wedge\tau_0} - \tilde{U}_{(t+\delta t)\wedge\tau_0}/\mathcal{F}_t) + \mathbb{I}_{t\geq\tau_0}\mathbb{E}(\tilde{U}_{t\wedge\tau_0} - \tilde{U}_{(t+\delta t)\wedge\tau_0}/\mathcal{F}_t)\\
&= \mathbb{E}(\mathbb{I}_{t<\tau_0}(\tilde{U}_{t\wedge\tau_0} - \tilde{U}_{(t+\delta t)\wedge\tau_0})/\mathcal{F}_t) + \mathbb{E}(\mathbb{I}_{t\geq\tau_0}(\tilde{U}_{t\wedge\tau_0} - \tilde{U}_{(t+\delta t)\wedge\tau_0})/\mathcal{F}_t)\\
&= \mathbb{E}(\mathbb{I}_{t<\tau_0}(\tilde{U}_t - \tilde{U}_{t+\delta t})/\mathcal{F}_t) + \mathbb{E}(\mathbb{I}_{t\geq\tau_0}(\tilde{U}_{\tau_0} - \tilde{U}_{\tau_0})/\mathcal{F}_t).
\end{aligned}$$

But, from (5.1), on $\{t < \tau_0\}$, we have $\tilde{U}_t = \mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t)$ and thus the first term is zero. This is of course also the case for the second term and thus $\tilde{U}_{t\wedge\tau_0}$ is indeed a martingale.

It follows that $\tilde{U}_{0\wedge\tau_0} = \mathbb{E}(\tilde{U}_{T\wedge\tau_0})$ and thus $U_0 = \mathbb{E}(\varphi(S_{\tau_0}))$.

We still have to check that for any $\mathcal{F}$-stopping time $\tau \in \mathcal{T}(t,T)$, $\mathbb{E}(\varphi(S_{\tau_0})) \leq \mathbb{E}(\varphi(S_\tau))$. Indeed, we have

$$\mathbb{E}(\varphi(S_{\tau_0})) = U_0 \geq \mathbb{E}(U_{t\wedge\tau}) = \mathbb{E}(U_\tau) \geq \mathbb{E}(\varphi(S_\tau))$$

the first inequality resulting from the fact that a stopped supermartingale (here $\tilde{U}_{t\wedge\tau}$) is still a supermartingale (exercise) and the second from the fact that, for any t we have from (5.1) that $U_t \geq \varphi(S_t)$. □

Remark: If, similarly to what we did for U_t, we denote by $\widetilde{\varphi(S_t)} := e^{-rt}\varphi(S_t)$ the discounted pay-off, formula (5.1) can just be written

$$\tilde{U}_t = \max\{\widetilde{\varphi(S_t)}, \mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t)\}.$$

We can now check that $\tilde{U}_t$ is a supermartingale, and more precisely that this formula defines it as the *smallest supermartingale that dominates the discounted pay-off* $\widetilde{\varphi(S_t)}$. This supermartingale is called the *Snell enveloppe* of this discounted pay-off $\widetilde{\varphi(S_t)}$.

[1]for any φ convex, $\mathbb{E}(\varphi(X)) \geq \varphi(\mathbb{E}(X))$, as the supergraph of φ is convex - observe where $\mathbb{E}(X, \varphi(X))$ and $(\mathbb{E}(X), \varphi(\mathbb{E}(X))$ are located ; in our finitary context, we have $\mathbb{E}(Y|\mathcal{F}_t)(\omega) = \mathbb{E}(Y|\overline{\omega}_t)$, where $\overline{\omega}_t$ is the atom of ω in the algebra $\mathcal{F}_t$.

5.3 Hedging strategy with consumption

We have built up the backward induction for the price of any american option as to be the smallest value for which the seller of the option can be hedged in any case, the holder exercising or not his right of early exercise. As we shall see it now, this is no longer an exact selffinanced hedge as in the european case, but a *superhedge* generally called *hedge with consumption*. Indeed, as lang as the exercise line has not been crossed, the premium U_0, invested in the hedging portfolio and dynamically managed just as for a the hedge of a european option, provides an exact hedge, as the value of the portfolio has at each time exactly the value of the american option. Once the exercise line has been crossed (if this happens) there are two possibilities. Either the holder of the option exercises his right of early exercise : he gets his pay-off and the options does no longer exist. Either he does not exercices his right (because he did not noticed that the exercise line has been crossed, either he has better things to do like getting married) and this case the seller can build up his hedging portfolio for a price stricly less than the pay-off, and he makes a profit on the expenses of the careless holder. This “income” will keep going as long as the stock price stays below the exercise line and that the holder does not require the early pay-off, thus creating a strictly positive wealth generally called “consumption” and that will stay with the seller of the option.

So the hedge of an american option is a *superhedging* that can either by a simple (exact) hedge, either can generate some consumption, depending on the behaviour of the stock and of the holder. There is a nice and elegant way to formalize this situation, using a result known as the *Doob-Meyer decomposition*.

Theorem 5.3 *Let $\tilde{U}_t$ be an $\mathcal{F}$-supermartingale. It exists a increasing and previsible random walk A_t (which means that for any t, A_t is $\mathcal{F}_{t-\delta t}$-mesurable) such that*

$$\tilde{U}_t = M_t - A_t,$$

where M is an $\mathcal{F}$-martingale. This decomposition of $\tilde{U}_t$ is called its Doob-Mayer decomposition.

Proof: The proof of this theorem is particularly simple in our discrete case. One defines the two walks A_t and M_t in the following way :

$$A_0 := 0 \quad A_{t+\delta t} := A_t + \mathbb{E}(\tilde{U}_t - \tilde{U}_{t+\delta t}/\mathcal{F}_t)$$

and

$$M_0 := 0 \quad M_{t+\delta t} := M_t + \tilde{U}_{t+\delta t} - \mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t)$$

Than we check that they have the required properties. First of all A_t is increasing as $\tilde{U}_t$ is a supermartingale, and it is previsible by construction. Moreover, one has :

$$\mathbb{E}(M_{t+\delta t} - M_t/\mathcal{F}_t) = \mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t) - \mathbb{E}(\mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t)/\mathcal{F}_t) = 0$$

just applying linearity and transitivity of conditional expectation. Thus M is a an $\mathcal{F}$-martingale. $\square$

In the following theorem, for the sake of simplicity, we replace by Z_t the discounted pay-off $e^{-rt}\varphi(S_t)$ of the american option.

Theorem 5.4 *Let (Z_t) be an $\mathcal{F}$-adapted process and let $\tilde{U}_t$ be its Snell envellope. Let $\tilde{U}_t = M_t - A_t$ be the Doob-Meyer decompostion of $\tilde{U}_t$. Then the optimal stopping time τ_0 defined by $\tau_0 = Min\{s \in [0..T]_{\delta t}\ ,\ U_s = Z_s\}$ is equal to the stopping time $\tau_A = Min\{s \in [0..T]_{\delta t}\ ,\ A_{s+\delta t} \neq 0\}$ if $A_T \neq 0$ and $\tau_0 = T$ else.*

This theorem states that the optimal stopping time, which means the optimal time for the holder to exercise, is the fist time when the increasing process A_t is no longer zero. This process thus turns out as being exactly the consumption. Indeed, as soon as the exercise line is crossed, if the holder does not exercise his right the value of the option is no longer the value of a selffinancing portfolio and begins to generate some consumption equal to the increasing process A_t. It is this consumption that is the reason why the american option is a supermartingale (and not just a martingale as in the case of a european option)

Proof: The proof shows successively that $\tau_A \geq \tau_0$ and that $\tau_A \leq \tau_0$.

- Let $t \in [0..T]_{\delta t}$. On $\{\tau_A = t\}$, $A_t = 0$ and $A_{t+\delta t} \neq 0$. Thus $\tilde{U}_t = M_t - A_t = M_t$ and $\tilde{U}_{t+\delta t} = M_{t+\delta t} - A_{t+\delta t} < M_{t+\delta t}$. Thus

$$\mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t) < \mathbb{E}(M_{t+\delta t}/\mathcal{F}_t) = M_t = \tilde{U}_t.$$

  As $\tilde{U}_t = \mathrm{Max}\{Z_t, \mathbb{E}(\tilde{U}_{t+\delta t}/\mathcal{F}_t)\}$, this implies $\tilde{U}_t = Z_t$. Thus, by définition of τ_0, $\tau_A \geq \tau_0$.

- Let $s \in [0..T]_{\delta t}$. On $\{\tau_0 = s + \delta t\}$, $\tilde{U}_{s+\delta t} = Z_{s+\delta t}$ and $\tilde{U}_s > Z_s$. Thus, as A_t is previsible,

$$\tilde{U}_s = \mathbb{E}(\tilde{U}_{s+\delta t}/\mathcal{F}_s) = \mathbb{E}(M_{s+\delta t} - A_{s+\delta t}/\mathcal{F}_s) = M_s - A_{s+\delta t} = \tilde{U}_s + A_s - A_{s+\delta t}$$

  Thus $A_{s+\delta t} = A_s$, and thus $\tau_A \leq \tau_0$.

□

Chapter 6

A stochastic interest-rates model

6.1 Some general facts on the present value of future money

When we speak on interests, we speak of the difference between future money and money right now. There is a difference between money and a loaf of bread: if you have a cup of rice or a loaf of bread and nobody needs it for a few days, it will get lost. Humanity has invented money to solve this problem, and as it is difficult to imagine that nobody would need this cup of rice, a technique has been invented as simple as possible to enter a contract in which somebody will exchange the present cup of rice in exchange of some, possibly future, "pay back". Money is supposed to be the simplest technique to do this. Lets now begin with a pleasant situation : you have one million and you do not need it right now. Then you are facing risks such as somebody could steal that money from you. So you should be prepared to accept to pay a certain amount to someone for keeping this money safely until some future day. This amount would be money that could be considered as the difference between present money and future money and is called *interest*. This would be, in that case, a negative interest. At the turn of the millenium, some Swiss banks would indeed impose negative interests if you wanted to make a deposit.

Of course this example is not common, and usually interests are positive. The fact that interests are not zero is not obvious at all. Interests are forbiden in the Jewish and Catholic religion, as far as it deals with loans between people of the same religion. A good Jew would only accept interests from non-Jews ; it is not clear to me if the Medicis family where considered good Catholics (for sure they had problems with the Pope of that time, but not for that reason). Moslems would not accept interests and it is still common experience that a Moslem would throw back to the banker the interests he would receive, not willing to lose his chances to enter Paradise against a handful of coins. To the contrary, this Moslem is welcome to enter a joint venture with others and to share the benefits ; this is why there are Moslem-oriented mutual funds that would be invested in stocks (and if there are some interests related with the cash held by the fund, they are given to some charity). Protestants had long discussions on the subject in Geneva, and they decided to accept interests, just as they would accept money for lending a field.

6.1.1 Where are the risks ?

Anyway, interests are now common practice and modern mathematical approach to interests is related to risks. One reason for the existence of different interest rates, it is because the risks can be different, and mathematical finance of interest-rates is related to no-arbitrage in this matter of risk : if there is an opportunity somewhere to take less risks for the same amount of interests, it will disappear immediately, as somebody will take advantage of it.

The main risk with loans lays in the "default risk" which means that the beneficiary of the loan can not pay back in due time what was decided. We will not consider here this question and will only consider the risks related to interests for sovereign debts, such as Treasury bonds of some reliable state. This could be for example buying, at $t_0 = 0$, for 950 some bond with *face value* of 1000 to be paid *at maturity* $T =$ one year, that is an interest of 50 for 950 ; as time moves and the maturity becomes closer, this bond will exchange at an increasing price, say 975 at some $t_1 < T$. Assume that at this date the same institution releases bonds with same face value of 1000 and same maturity T, for 974. Immediately nobody would buy the previous bond for 975, but only for 974. This would mean that the interests would increase from 25 for 975 to 26 for 974 (for a time-to-maturity of $T - t_1$) and explains how the value of a bond faces a risk, as the releasing institution has the full right to sell for less (or more) future money. This also shows why an upward change of interest-rates results in an immediate downward change of bond values.

The same institution may release at time 0 a bond with face value 1000 with maturity $2T$, at a price that does not need to correspond to the same compound interest-rate, as the two bonds do not correspond to real money at the same date (one is 1000 at T and the second is 1000 at $2T$; you would buy one or the other according to when you need to get 1000 or on what your anticipations are on what the interest-rates will be at time T for the maturity $2T$).

This is how interest-rates depend on time t and maturity T. If $t = 0$ (always considered to be the present time) these interest-rates are known, as a function of T. As soon as $t > 0$, they are not, as the souvereign releasing institution may change its interests rates in the interval, and it is common sense to introduce stochastic process to models. Observe that for each t, one does not have just one number but a complete function of the maturity T called term-structure at time t, for $t < T$.

6.1.2 Zero coupons and the term structure

There are different ways of expressing interest-rates. We introduce some of them here.

Definition: A *zero-coupon bond with maturity* T is an asset that will pay 1 (its *face value)* at time T. Its value at time $t \in [0, T]$ (when the *time-to-maturity* will be $\theta := T - t$) is denoted by $Z(t, T)$, so $Z(T, T) = 1$.

Usually a bond would pay a *principal* P at maturity T and *coupons* $c_1, \ldots, c_n$ at time $t_1 < \ldots < t_n$. A zero-coupon is thus a convenient conceptual asset that allows to express the value at anytime t of any real bond as $PZ(t,T) + \sum_{t_i \in (t,T]} c_i Z(t,T)$.

In elementary (nonrandom) financial mathematics, one would usually consider a compound interest-rate a, such that a deposit of 1 at time $t = 0$ would have a value of $(1 + a)^T$ at time T. To this rate, in the mathematical finance setting, corresponds the *actuarial rate* (or *effective rate*) that is the function $(t, T) \mapsto a(t, T)$ such that

$$Z(t, T) = (1 + a(t, T))^{-(T-t)}.$$

In financial mathematics, one often prefers to consider the countinuously compound interest-rate r, such that a deposit of 1 at time $t = 0$ would have a value of e^{rT} at time T. This becomes the *yield-to-maturity* Y in this setting, which is the function $(t, T) \mapsto Y(t, T)$ such that

$$Z(t, T) = e^{-Y(t,T)(T-t)}.$$

The *instantaneous forward-rate* is the function $(t, T) \mapsto f(t, T)$ such that

$$Z(t, T) = e^{-\int_t^T f(t,u)du}.$$

This is the limiting case of the *forward-rate* $(t, T, U) \mapsto f(t, T, U)$, $t \le T < U$, as $U \to T^+$, that is the function that satifies

$$Z(t, U) = Z(t, T)e^{-f(t,T,U)(U-T)}, \text{ for } t < T < U.$$

Observe that these are relations between various ways of expressing the interest-rates. The *term-structure of* (souvereign) *interest-rates* or *yield curve* is the function $T \mapsto Y(t,T)$. For $t = 0$ it is observed on the market ; for $t > 0$ it is a random curve (a r.v. with values in a set of functions).

6.1.3 Short-term interest-rates and actualization

In any market, there is a short-term interest rate in force, at which the daily settlement between the various traders on the market takes place for overnight debts between them. This fixes the costs of money for the traders and, in the context of random interest-rates, actualization is related to that short-term (overnight) interest-rate denoted by $a_t = a(t - \delta t, t) \in \mathcal{F}_{t-\delta t}$. From the point of view of mathematical models, it is random and actualization is performed using

$$B_t = (1 + a_{\delta t})(1 + a_{2\delta t}) \ldots (1 + a_t),$$

or equivalently $\frac{1}{B_t} = \frac{1}{1+a_{\delta t}} \frac{1}{1+a_{2\delta t}} \cdots \frac{1}{1+a_t} = Z(0, \delta t)Z(\delta t, 2\delta t) \ldots Z(t - \delta t, t)$. Specifically, $\tilde{X}_t = X_t/B_t$. One can think of B_t as the random value of a saving account for which you have an initial deposit of 1 and bearing (compound) interests day after day on the basis of the observed random short-term interest-rate. Even if random, B_t is known as the *riskless interest-rate*, as traders know the (daily) short-term rate when they enter any transaction.

6.1.4 Stochastic rollover

In the case of deterministic interest-rates, for any $s \le t \le u$ we must have $Z(s,t)Z(t,u) = Z(s,u)$ as it is easy to see that the left-hand side is exactly the amount to invest at time s to have the right amount at time t in order to get finally 1 at time u, and this is just the characteristic value of the right-hand side $Z(s,u)$. In case of stochastic interest-rates $Z(s,t)$ and $Z(s,u)$ are already known at time s, whereas $Z(t,u)$ will be only known at time $t \ge s$. So, for any stochastic model, it is useful to define the r.v. $\eta = \eta(s,t,u)$ such that

$$Z(t,u) = \frac{Z(s,u)}{Z(s,t)} \eta(s,t,u). \tag{6.1}$$

6.2 The Ho and Lee model for the term structure

We want to introduce now the equivalent, for interest-rates, of the Cox-Ross-Rubinstein model for a stock, namely the Ho and Lee binomial model. As it is an interest-rates model, its main particularity is that its values are not numbers but *curves* $T \mapsto Z_t(T) = Z(t,T)$, for $t \le T \le T_{\max}$, $t \in \mathbb{T} := [0..T_{\max}]_{\delta t}$, $\delta t := T_{\max}/N$ for some fixed N. This will be achieved by choosing conveniently a deterministic function $(\theta, x) \mapsto \eta(\theta, x)$ such that

$$Z^T_{t+\delta t} = \frac{Z^T_t}{Z^{t+\delta t}_t} \eta(\theta^T(t+\delta t), X_{t+\delta t}), \tag{6.2}$$

where $Z^T_t := Z(t,T)$, and $\theta^T(s) := T - s$ is the *time-to-maturity.*

The similarity with the Cox-Ross-Rubinstein model is that $Z_{i\delta t}$ takes only $i+1$ values, depending on the value $j = J_i(\omega)$, with $J_i = \delta J_1 + \ldots + \delta J_i$, where the $(\delta J_i)_{i \ge 1}$ are independent and identically distributed Bernoulli r.v.. In other words $\delta J_i \rightsquigarrow \mathcal{B}(\pi_i, 1)$ (by no-arbitrage, it will soon turn out that the π_i must all be equal). More precisely, we define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and for $k \ge 1$, let $\mathcal{F}_{k\delta t} = \sigma(\delta J_1, \ldots, \delta J_k) = \sigma(X_{\delta t}, \ldots, X_{k\delta t})$, with $X_{i\delta t} = \delta J_i$. The random *functions* $Z_t : [t..T_{\max}] \to \mathbb{R}^+$, $T \mapsto Z^T_t$, will be chosen such that they are $\mathcal{F}_t$-measurable, and even $\sigma(J_i)$-measurable for $t = i\delta t$. As mentioned, the curve $Z_t(\omega)$ should depend only on the value $j = J_i(\omega) = \delta J_1(\omega) + \ldots + \delta J_i(\omega)$ and not of the specific values of $\delta J_1(\omega)$, ..., $\delta J_i(\omega)$. It is in this sens that the model will be *binomial.*

We will now build up the Ho and Lee model, determining the necessary form of the π_i and η.

6.3 The model as a three-parameters model : π, δ, and N

6.3.1 No arbitrage condition

Our first concern is of course no arbitrage. For each $T \in \mathbb{T}$, the actualized value of Z_t^T should be a martingale for some probability $\mathbb{P}^*$. The only choice we have for $\mathbb{P}^*$ is the Bernoulli laws of the δJ_i random variables (independent by assumption), so we introduce the parameters

$$\pi_i := \mathbb{P}^*(\{\delta J_i = 0\}) = \mathbb{P}^*(\{X_{t_i} = 0\}).$$

Now, in view of Theorem 4.2, and Proposition 3.1 applied to $S_t := Z_t^T$, we will have no arbitrage if and only if

$$Z_t^T = \mathbb{E}^*(Z_t^{t+\delta t} Z_{t+\delta t}^T \mid \mathcal{F}_t)$$

for any $t \in [0..T)$. Using (6.2), we obtain

$$\begin{aligned} Z_t^T &= \mathbb{E}^*(Z_t^{t+\delta t} Z_{t+\delta t}^T \mid \mathcal{F}_t) \\ &= \mathbb{E}^*(Z_t^T \eta(\theta^T(t+\delta t), X_{t+\delta t}) \mid \mathcal{F}_t) \\ &= Z_t^T \mathbb{E}^*(\eta(\theta^T(t+\delta t), X_{t+\delta t}) \mid \mathcal{F}_t) \\ &= Z_t^T \mathbb{E}^*(\eta(\theta^T(t+\delta t), X_{t+\delta t})) \text{ as } X_{t+\delta t} \text{ is independent of } \mathcal{F}_t \end{aligned}$$

So, dividing by Z_t^T that is positive, we get

$$1 = \pi_i \eta(\theta, 0) + (1 - \pi_i)\eta(\theta, 1) \tag{6.3}$$

for any $\theta = \theta^T(t + \delta t) \in (0..T_{\max}]_{\delta t}$, so $\pi_i = (1 - \eta(\theta, 1))/(\eta(\theta, 0) - \eta(\theta, 1))$ can't change with i and must have a constant value π. Moreover, using (6.2) for $t' := T - \delta t$, we have $\theta^T(t' + \delta t) = \theta^T(T) = 0$, and

$$1 = Z_T^T = Z_{t'+\delta t}^T = \frac{Z_{t'}^T}{Z_{t'}^{t'+\delta t}} \eta(\theta^T(t' + \delta t), X_{t'+\delta t}) = \frac{Z_{T-\delta t}^T}{Z_{T-\delta t}^{T-\delta t+\delta t}} \eta(\theta^T(T), X_T) \text{ for } X_T \in \{0, 1\} \text{ , so}$$

$$\eta(0, x) = 1 \text{ for any } x \in \{0, 1\}. \tag{6.4}$$

Thus, we have shown :

Proposition 6.1 *Any interest-rate model satisfying (6.2), with $X_{t_i} \rightsquigarrow \mathcal{B}(\pi_i, 1)$ independent Bernoulli random variables is arbitrage-free if and only if $\eta(0, x) = 1$ for any $x \in \{0, 1\}$, all the π_i are equal and their common value π satisfies*

$$1 = \pi\eta(\theta, 0) + (1 - \pi)\eta(\theta, 1). \tag{6.5}$$

6.3.2 Binomial condition

Now, using the fact that, for $t = i\delta t$, Z_t^T depends only on J_i, we have

Lemma 6.2 *Under the no-arbitrage condition (6.5), for any $\theta \in [0..T)_{\delta t}$, following identity holds :*

$$\eta(\theta + \delta t, 1)\eta(\theta, 0)\eta(\delta t, 0) = \eta(\theta + \delta t, 0)\eta(\theta, 1)\eta(\delta t, 1), \tag{6.6}$$

and thus

$$\eta(\theta, 0) = \frac{1}{\pi + (1 - \pi)\delta^{\frac{\theta}{\delta t}}} \text{ and } \eta(\theta, 1) = \delta^{\frac{\theta}{\delta t}} \eta(\theta, 0) \text{ , with } \delta := \frac{\eta(\delta t, 1)}{\eta(\delta t, 0)} > 1. \tag{6.7}$$

Proof: Formula (6.7) is the consequence of the fact that the model should be binomial, that is, for $t = i\delta t$, Z_t^T should depend only on $j = J_i(\omega)$ and not of the specific values of $\delta J_1(\omega)$, $\ldots$, $\delta J_i(w)$ that add up to form $J_i(\omega)$. This will be true if and only if the tree is *recombining*, which means that an *up* followed by a *down* should give the same result as a *down* followed by an *up*. In other words, if we have two $\omega' \in \Omega$ and $\omega'' \in \Omega$ such that $J_i(\omega') = J_i(\omega'') = j$ and $J_{i+2}(\omega') = J_{i+2}(\omega'') = j+1$, but $\delta J_{i+1}(\omega') = 1$ and $\delta J_{i+2}(\omega') = 0$, whereas $\delta J_{i+1}(\omega'') = 0$ and $\delta J_{i+2}(\omega'') = 1$, the values of $Z_{i\delta t}^T$ and $Z_{(i+2)\delta t}^T$ should not depend on whether $\omega = \omega'$ or $\omega = \omega''$. Applying (6.2) twice, we have

$$\begin{aligned}
Z_{t+2\delta t}^T &= \frac{Z_{t+\delta t}^T}{Z_{t+\delta t}^{t+2\delta t}}\eta(\theta^T(t+2\delta t), X_{t+2\delta t}) \\
&= \frac{Z_t^T}{Z_t^{t+\delta t} Z_{t+\delta t}^{t+2\delta t}}\eta(\theta^T(t+\delta t), X_{t+\delta t})\eta(\theta^T(t+2\delta t), X_{t+2\delta t}) \\
&= \frac{Z_t^T}{Z_t^{t+2\delta t}\eta(\theta^{t+2\delta t}(t+\delta t), X_{t+\delta t})}\eta(\theta^T(t+\delta t), X_{t+\delta t})\eta(\theta^T(t+2\delta t), X_{t+2\delta t}) \text{ , again by (6.2)} \\
&= \frac{Z_t^T}{Z_t^{t+2\delta t}}\frac{\eta(\theta+\delta t, X_{t+\delta t})\eta(\theta, X_{t+2\delta t})}{\eta(\delta t, X_{t+\delta t})} \text{ , for } \theta := \theta^T(t+2\delta t).
\end{aligned}$$

Now, as $J_i(\omega') = J_i(\omega'')$ and $J_{i+2}(\omega') = J_{i+2}(\omega'')$, $Z_{t+2\delta t}^T$, Z_t^T, and $Z_t^{t+2\delta t}$ do not depend on whether $\omega = \omega'$ or $\omega = \omega''$, so neither does

$$\frac{\eta(\theta+\delta t, X_{t+\delta t})\eta(\theta, X_{t+2\delta t})}{\eta(\delta t, X_{t+\delta t})}.$$

So, equating the two values obtained for $\omega = \omega'$ and $\omega = \omega''$, we get

$$\frac{\eta(\theta+\delta t, 1)\eta(\theta, 0)}{\eta(\delta t, 1)} = \frac{\eta(\theta+\delta t, 0)\eta(\theta, 1)}{\eta(\delta t, 0)},$$

which implies (6.6).

Now, from (6.5) we have

$$\eta(\theta, 1) = \frac{1}{1-\pi}(1 - \pi\eta(\theta, 0)), \tag{6.8}$$

so (6.2) becomes

$$\frac{1}{1-\pi}(1-\pi\eta(\theta+\delta t,0)\eta(\theta,0)\eta(\delta t,0) = \frac{1}{(1-\pi)^2}\eta(\theta+\delta t,0)(1-\pi\eta(\theta,0))(1-\pi\eta(\delta t,0)). \tag{6.9}$$

Letting $x_n = \frac{1}{\eta(\theta,0)}$, $x_{n+1} = \frac{1}{\eta(\theta+\delta t,0)}$, thus $x_1 = \frac{1}{\eta(\delta t,0)}$, (6.9) becomes

$$(1-\pi)\left(1-\frac{\pi}{x_{n+1}}\right)\frac{1}{x_n}\frac{1}{x_1} = \frac{1}{x_{n+1}}\left(1-\frac{\pi}{x_n}\right)\left(1-\frac{\pi}{x_1}\right).$$

Multiplying both sides by $x_1 x_n x_{n+1}$ we get $(1-\pi)(x_{n+1}-\pi) = (x_n-\pi)(x_1-\pi)$, or, equivalently

$$x_{n+1} = \pi + \frac{1}{1-\pi}(x_n-\pi)(x_1-\pi) =: x_n\delta + \gamma,$$

with $\delta = \frac{x_1-\pi}{1-\pi}$ and $\gamma = \pi - \frac{\pi}{1-\pi}(x_1-\pi) = \pi(1-\delta)$. Using $x_1 = \frac{1}{\eta(\delta t,0)}$ we get $\eta(\delta t, 0) = \frac{1}{\pi+(1-\pi)\delta}$. Now, as $1 = \pi\eta(\delta t, 0) + (1-\pi)\eta(\delta t, 1)$,

$$\delta = \frac{1}{1-\pi}\left(\frac{1}{\eta(\delta t,0)} - \pi\right) = \frac{1}{1-\pi}\frac{1-\pi\eta(\delta t,0)}{\eta(\delta t,0)} = \frac{\eta(\delta t,1)}{\eta(\delta t,0)}.$$

Finally, solving $x_n = x_{n-1}\delta + \pi(1-\delta)$ one gets $x_n = (1-\pi)\delta^n + \pi$, so

$$\eta(\theta, 0) = \eta(n\delta t, 0) = \frac{1}{x_n} = \frac{1}{\pi + (1-\pi)\delta^n} = \frac{1}{\pi + (1-\pi)\delta^{\frac{\theta}{\delta t}}},$$

and, using (6.8),

$$\eta(\theta, 1) = \frac{1}{1-\pi} - \frac{\pi}{\pi + (1-\pi)\delta^{\frac{\theta}{\delta t}}} = \frac{\delta^{\frac{\theta}{\delta t}}}{\pi - (1-\pi)\delta^{\frac{\theta}{\delta t}}} = \delta^{\frac{\theta}{\delta t}}\eta(\theta, 0).$$

□

Appendix A

Exercises with Maple

A.1 The net $\varphi(t, S)$ and the delta-hedge in a binomial tree model

Let us consider a Cox-Ross-Rubinstein model $S_{t+\delta t} = S_t U_{t+\delta t}$, with $U_s \in \{u, d\}$, $u = e^{+\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}} (= \frac{1}{u})$.

1. The program below allows to plot the binomial tree (t, S_t) for a Cox-Ross-Rubinstein model, with initial value $S_0 = 140$ and volatility $\sigma = 0,4$.

   In order to become familiar with some usefull Maple instructions, read this program run it and study its various instructions, using the online "help".

2. Using ideas from this program, plot the graph of the function $(t, S_t) \mapsto \varphi(t, S_t) := C_t$, that is defined on the previous binomila tree, for an at-the-money call ($K = S_0$), that is defined by backward induction from its final value $C_T = (S_T - K)^+$, and the formula $C_{t-\delta t} = e^{-r\delta t}\mathbb{E}^*_{t-\delta t}(C_t)$, for the probability $\mathbb{P}^*$ for which the discounted underlying asset is a martingale, caracterized by $p = \frac{R-d}{u-d}$, where $R = e^{r\delta t}$; (actually, we'll assume here that $r = 0$ and thus $R = 1$). Your result will be a 3-dimensional object ; clicking at the picture will allow the picture to show up on the toolbar ; click on the tool that produces a box arround your "Call_net" ; pressing on the left button allows you to "seize" the box and turn it around (you could have programmed this by adding `axis=BOXED` in your plot instruction ; try `?plot(options) ; .`)

3. Same exercise for a put option P_t on the same underlying asset, and with exercise price $K = 130$.

4. Plot on a same graph the curves $s \mapsto \varphi(t, s)$ for some final values of t : $t = T$, $t = T - \delta t$, $t = T - 2\delta t$, ..., $t = T - 5\delta t$, in order to see how the price evolves close to the exercise date, as a function of S_t. This can be achived by pieces like

   `line([i,S(i,j),C(i,j)],[i,S(i,j+1),C(i,j+1)])` in order to attache together point corresponding to the same $t = \mathtt{i}\delta t$.

5. Same exercise for the function $S_t \to \Delta_t(t, S_t)$ giving de "delta hedge ratio" $x = \Delta_t = \frac{\Pi_t^+ - \Pi_t^-}{S_t^+ - S_t^-}$.

Maple programme: Computation and plotting of the underlying tree in a Cox-Ross-Rubinstein model

```
> restart;
```

```
> n:=10:T:=1:delta_t:=evalf(T/n):
```

```
> S0:=140:sigma:=0.4:
> up:=exp(sigma*sqrt(delta_t)):
> down:=exp(-sigma*sqrt(delta_t)):
> S:=proc(i,j) option remember:
> if i=0 then S0:
> elif j=0 then evalf(S(i-1,0)*down):
> else evalf(S(i-1,j-1)*up) fi:  end:
> with(plottools):
> tree:=proc(i,j) option remember:
> if i<n then line([i,S(i,j)],[i+1,S(i+1,j)]),tree(i+1,j),
> line([i,S(i,j)],[i+1,S(i+1,j+1)]),tree(i+1,j +1)
> fi:end:
> plots[display](tree(0,0));
```

A.2 American options

Up to here, we have only considered european option, that is, options that give the right to "something" at only one time, namely the exercice time T. Actually, options are usualy "american options", which means that this right is granted (once) at *any* time t *until* T. So, there are more rights related with an american option, and thus it is reasonable to anticipate that the price of an american option is higher than the price of the corresponding european option, or at least not smaller. We will see that the price of an american put is indeed higher than a european put, but that the price of an american call is equal to the price of a european call.

A.2.1 The dynamic programming approach

Let us begin with the approach we used for the european options in the Cox-Ross-Rubinstein model $S_{i\delta t} = S_0 u^{J_{i\delta t}} d^{i-J_{i\delta t}}$, using the risk-neutral probability $\mathbb{P}^*$ for which $\mathbb{P}^*\{\delta J_{i\delta t} = 1\} = \frac{R-d}{u-d}$. Let φ be the payoff function of the considered american option. So the writter of the option is untitled to ask for $\varphi(S_t)$ at one time $t \in (0..T]_{\delta t}$ of his choice.

In order to hedge this option, we build a hedging portfolio consisting at time t in α_t stocks and $\beta_t e^{r\delta t}$ cash. Let X_t be at each time the minimal value of this portfolio. Now, as we consider an american option, we have to face two problems at time t : the value X_t of the portfolio should be sufficient to build up the correct portfolio $(\alpha_{t+\delta t}, \beta_{t+\delta t})$ for the "next day" (i.e. next time step) and, as similarly to what we explained for the european option, this requires

$$X_t \geq e^{-r\delta t}\mathbb{E}^*(e^{-r\delta t} X_{t+\delta t} | S_t), \tag{A.1}$$

but also, now, as the owner may exercise "today", the worth of the hedge portfolio has also to satisfy

$$X_t \geq \varphi(S_t). \tag{A.2}$$

So, as we are looking for the portfolio of minimal price, putting (A.1) and (A.2) together, we get the following *american option backward-induction formula* :

$$\begin{aligned} X_T &= \varphi(S_T), & \text{(A.3)} \\ X_t &= \mathrm{Max}\,(\varphi(S_t), \mathbb{E}^*(e^{-r\delta t} X_{t+\delta t} | S_t)). & \text{(A.4)} \end{aligned}$$

A.2.2 Programming

Let us consider the usual Cox-Ross-Rubinstein model $S_{t+\delta t} = S_t U_{t+\delta t}$, with $U_s \in \{u, d\}$, $u = e^{+\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}} (= \frac{1}{u})$.

We assume that you have access to the constants, procedures, and objects defined in the "Exercise #1" such as `S0`, `n`, `sigma`, `S(i,j)`, `Call(i,j)` (for the european call option), `Put(i,j)`, `tree(i,j)` (for `Calltree(i,j)`), `Puttree(i,j)`, `Callnet(i,j)` (for the 3-dimensionnal graph of `(i,S(i,j)`$\mapsto$`Call(i,j))`, and `Putnet(i,j)`. Here it is essential that $r \neq 0$, for instence $r = 0.05 = 5\%$. Define `Call` and `Put` accordingly.

1. Define a procedure `CallAmer(i,j)` giving the value X_t of the american call-option at time $t =$`i*deltat_t`, when $S_t =$`S(i,j)`.

2. Check that `CallAmer(i,j)-Call(i,j)` is 0 for `(i,j)`$= (0,0)$ and a few other values.

3. Define a procedure `PutAmer(i,j)` giving the value of the american put-option at time $t =$`i*deltat_t`, when $S_t =$`S(i,j)`.

4. Check that `PutAmer(i,j)-Put(i,j)` is strictly positive for `(i,j)`$= (0,0)$ and a few other values ; is it true for any `(i,j)` ? (**hint :** check for `i=n` or `j`$= 0$)

5. Define a procedure `PutAmernet(i,j)` similar to `Putnet(i,j)` giving the 3-dimensionnal graph of `(i,S(i,j)`$\mapsto$`PutAmer(i,j))`. Display `Putnet(0,0)` and `PutAmernet(0,0)` on the same picture ; compare the values of european and american put-option.

6. Define a procedure `treeAmer(i,j)` similar to `tree(i,j)`, but for which the points `(i,S(i,j))` are displayed in different colors et symbols according to whether `PutAmer(i,j)=PutPayOff(i,j)` or `PutAmer(i,j)<PutPayOff(i,j)`. Observe that `PutAmer(i,j)=PutPayOff(i,j)` under some line called *exercise line.*

A.3 Convergence of the CRR price towards the Black-Scholes price

1. Explore the *finance* library of Maple questionning `?finance` and studying the

   `blackscholes(S,K,r,T,sigma,'hedge')`

   procedure. This command has six parameters : the initial (present) price `S`$= S_0$ of the underlying asset, the exercise (strike) price `K` of the call-option, the constant riskless interest rate `r`, the exercise date `T` (in the unit for which the interest is e^r), the volatility `sigma`$= \sigma$, and the `hedge`, which is the name chosen for some Maple variable to which the command will give the *hedge-ratio* Δ as value that the command will have computed by the way. Try out the command `blackscholes(S,K,r,T,sigma,'hedge')` ;that will give you the Black-Scholes formula with "abstract" values `S,K,r,T,` and `sigma`, and again with

   $$S_0 = 140 \ , \ K = 160 \ , \ r = 0.25 \ , \ T = 1, \sigma = 0.4, hedge =' hedge')$$

   .

2. Check experimentaly, choosing various values of σ, that the Black-Scholes price BS is a strictly monotonous function of σ. Is it increasing or decreasing ? You may wish to plot your result for $\sigma \in [0.05..0.50]$.

3. Using the function `binomial` of Maple, plot, for $n = 10$, $n = 20$, $n = 50$ the histograms (use `histogram`) of the binomial probability law $\mathcal{B}(n, p)$ for $p = 0.55$.

4. Compute, using the *Cox-Ross-Rubinstein exact formula*, the CRR price

$$\mathrm{CRR} = S_0 \mathbb{E}_q(\mathbb{I}_{\{S_T > K\}}) - e^{-rT} K \mathbb{E}_p(\mathbb{I}_{\{S_T > K\}}) \ , \text{ with } p = \frac{R-d}{u-d} \text{ and } q = \frac{up}{R}.$$

for the values above of the parameters, choosing for example $n = 10$, 50, 100, and the model $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}}$ (and $R = e^{r\delta t}$).

5. Plot the CRR price as a function of n for $n = 10..100$ together with the limit value BS and see how the convergence goes.

6. Same question in the case $K = 140$ (*at-the-money call*).

A.4 The Ho and Lee model for interest-rates, and interest-rates derivatives

The Ho and Lee model is a model for the value of a zero-coupon Z_t^T, $t, T \in [0..T_{\max}]_{\delta t} =: \mathbb{T}$, $\delta t := T_{\max}/N$, $t \leq T$, where Z_t^T is the value, at time t of a contract paying 1 at time T. So $Z_T^T = 1$ for any $T \in \mathbb{T}$. It is a stochastic model defined on a set Ω allowing to code all the "states of the worlds" taken into account by the model, filtered by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ allowing to code the information availble at time $t \in \mathbb{T}$. Actually, in this model, the only relevant information is contained in the sequence of the values of the r.v. $(X_t)_{t\in\mathbb{T}^*}$, $\mathbb{T}^* :=]0..T]_{\delta t}$, $X_t \in \{0,1\}$, all the X_t sharing the same (Bernoulli) law, with $\mathbb{P}^*(X_t = 0) = \pi$ and $\mathbb{P}^*(X_t = 1) = 1 - \pi$, $\mathcal{F}_t$-mesurable, and independent of $\mathcal{F}_{t-\delta t}$[1]. For all $t \in \mathbb{T}^*$, let $J_t := \sum_{s\in]0..t]_{\delta t}} X_s$. Denote by i and k, $i \leq k$, the integers such that $t = i\delta t$ and $T = k\delta t$; the caracteristic property of a Ho and Lee model is that the $Z_t^T(\omega)$ belong to a binary (recombining) tree, which means that $Z_{i\delta t}^T(\omega)$ takes only $i+1$ different values depending only on $j = J_{i\delta t}(\omega)$ ($Z_{i\delta t}^T$ is $\sigma(J_{i\delta t})$-measurable). For $0 \leq j (= J_{i\delta t}(\omega)) \leq i \leq k$, we will write $Z_{i\delta t}^{k\delta t}(\omega)$:=`Z(i,j,k)`.

1. Show that `Z(k,j,k)`$= 1$.

2. We have shown that any Ho and Lee model is arbitrage-free and satisfies :

$$Z_t^T = \frac{Z_{t-\delta t}^T}{Z_{t-\delta t}^t} \eta(\theta^T(t), X_t) \ , \text{ with } \theta^T(t) := T - t, \tag{A.5}$$

for a function η defined by the choice[2] of a $\delta > 1$, that caracterises, together with $\pi \in]0,1[$ and the number of time-steps N, the chosen model, defined by

$$\eta(\theta, 0) := \frac{1}{\pi + (1-\pi)\delta^{\frac{\theta}{\delta t}}} \text{ and } \eta(\theta, 1) = \eta(\theta, 0) \cdot \delta^{\frac{\theta}{\delta t}} \tag{A.6}$$

The values of the Z_0^T, $T \in \mathbb{T}$, may be chosen in an arbitrary way (from the no-arbitrage point of view), allowing to adjust them to the spot values of the zero-coupons on the market. Check that $\mathbb{E}^*(\eta(\theta, X_t) \mid \mathcal{F}_{t-\delta t}) = \mathbb{E}^*(\eta(\theta, X_t)) = 1$, for all θ and t in $\mathbb{T}^*$, where $\mathbb{E}^*$ stands for the expectation with respect to the arbitrage probability $\mathbb{P}^*$.

3. The risk-less short rate, denoted by r_t, is defined by

$$(1 + r_{t+\delta t}) Z_t^{t+\delta t} = 1 \ ; \tag{A.7}$$

so we see that $r_{t+\delta t}$ is $\mathcal{F}_t$-mesurable (the process $(r_t)_{t\in\mathbb{T}^*}$ is $\mathbb{F}$-*predictable*) ; let

$$B_t := (1 + r_{\delta t})(1 + r_{2\delta t}) \ldots (1 + r_t) \text{ and } \tilde{Z}_t^T := Z_t^T / B_t.$$

[1] Beware, in the paper of M. Leippold and Z. Wiener, they choose $\mathbb{P}(X_t = 0) = (1 - \pi)$.

[2] It is the choice $\delta (= \eta(\delta t, 1)/\eta(\delta t, 0)) > 1$ that expresses that having $X_t(\omega) = 1$ codes an "*up*" and $X_t(\omega) = 0$ codes a "*down*".

By showing that $\mathbb{E}^*(\tilde{Z}_t^T \mid \mathcal{F}_{t-\delta t}) = \tilde{Z}_{t-\delta t}^T$, prove that $(\tilde{Z}_t^T)_{t\in\mathbb{T}}$ is a $(\mathbb{F},\mathbb{P}^*)$-martingale, and thus, that the model is arbitrage-free.

4. Here an implementation of the model for which one has $T_{\max} = N$ (and thus $\delta t =$ 1=`delta_t`), t =`i*delta_t`, T =`k*delta_t`, $J_t(\omega)$ =`j`, $T-t$ =`l*delta_t`, T =`k*deltat_t`, $\eta(T-t, X_t(\omega))$ =`eta(l*delta_t,x)`, for `x`= $X_t(\omega)$, $Z_t^T(\omega)$ =`Z(i,j,k)`, for $J_t(\omega)$ =`j`, with a choice π =`pi:=0.5`, and δ =`delta:=1.01`.

```
> restart:with(plots):with(plottools):
> N:=30:  Tmax:=N: delta_t:=Tmax/N: pi:=0.5:delta:=1.01:
> r:=0.025:# taux-court initial
> Z0:=proc(k) option remember; (1+r)^(-k) end:  # T=k*delta_t
> StructureParTermesInitiale:=[seq( [k,Z0(k)], k=1..N )]:
> plot(StructureParTermesInitiale):
> eta:=proc(l,x) option remember; # T-t=l*delta_t
> if x=0 then 1/(pi+(1-pi)*delta^(l*delta_t)) else eta(l,0)*delta^l
fi end:
> Z:=proc(i,j,k) option remember; #t=i*delta_t et T=k*delta_t
> if k<i then 1 # k<i n'a pas de sens, mais on veut pouvoir dessiner
> elif i=0 then Z0(k)
> elif j>0 then Z(i-1,j-1,k)/Z(i-1,j-1,i)*eta(k-i,1)
> else Z(i-1,j,k)/Z(i-1,j,i)*eta(k-i,0) #ici i>0 et j=0
> fi
> end:
> BrancheZ:=proc(i,j,k) option remember;
> if i<k then line([i,Z(i,j,k)],[i+1,Z(i+1,j,k)],color=blue),
> BrancheZ(i+1,j,k),
> line([i,Z(i,j,k)],[i+1,Z(i+1,j+1,k)],color=red,linestyle=3),
> BrancheZ(i+1,j+1,k)
> fi end:
> ArbreZ:=proc(k) option remember;
> BrancheZ(0,0,k),labels=['t','Z'],thickness=3 end:
> plots[display](ArbreZ(8)):
```

(a) What is the chosen function $T \mapsto Z_0^T$ which stands for the present values of the zero-coupons Z_t^T ?

(b) Practice your understanding of the values of Z^8 : what is the value of Z_8^8 ? What is the value of Z_0^8 and find this value on the curve `StructureParTermesInitiale` ? What is the value of Z_4^8 after two "*ups*" and two "*downs*" ? What is the value of Z_6^8 after only "*ups*" ? In this last case the zero-coupon that matures at $T = 8$ is said to be "*above par*" ; why does the existence of such an issue in the model seems sometime to be a draw-back of this model ?

5. **Yield :** The Yield of a zero-coupon is the rate denoted by Y_t^T such that

$$Z_t^T(1+Y_t^T)^{\frac{T-t}{\delta t}} = 1.$$

It is thus defined only for $t < T$.

(a) Define a procedure `Y(i,j,k)` corresponding to the yield of a zero-coupon $Z_{i\delta t}^{k\delta t}(\omega)$ when $J_{i\delta t}(\omega) = j$ and that has value `Z(i,j,k)`.

(b) Plot the tree of the yiels connecting each value of Y_t^T the two possible values $Y_{t+\delta t}^T$ that may come next in this model.

(c) What is the sign here of what you observed for Z_6^8 in the previous question.

6. **Caplets and Caps :** When subscribing a loan with changing rates, one may subscribe a contract that would take in charge the excess of interests to be payed above a maximal rate K. Typically, if the interest r_T to be payed at time T for a loan of one at time $T-\delta t$, this contract will pay $(r_T - K)^+$. This contract is called a *caplet* maturing at T with ceiling K. For a loan of one to be payed at time $T_{\max}$ and interests to be payed after each δt = one year, one should subscribe a *Cap*, which it the sum of all caplets maturing at $T \in]0..T_{\max}]_{\delta t}$. As the Ho and Lee model is a binary model, an interest rate derivative such as a caplet may be hedged, at time $t-\delta t$, by a portfolio bearing both riskless $Z_{t-\delta t}^t$ and risky $Z_{t-\delta t}^{t+\delta t}$. Its value and the quantity to hold can be computed similarly to options in a (binary) Cox-Ross-Rubinstein model, and as the processes $(\tilde{Z}_t^T)_{t\in[0..T]}$ are, for any $T \in \mathbb{T}$, $(\mathbb{F}, \mathbb{P}^*)$-martingales, one gets, for the value of the hedge portfolio

$$\text{Caplet}_{t-\delta t}^T(K) = \mathbb{E}^*(\text{Caplet}_t^T(K) \mid \mathcal{F}_{t-\delta t})/(1+r_t) \tag{A.8}$$

(and, more generally, for any $s \le t$, $\text{Caplet}_s^T(K) = \mathbb{E}^*\left(\text{Caplet}_t^T(K)\frac{B_s}{B_t} \mid \mathcal{F}_s\right)$). Similarly to the case of zero-coupons and yiels, denote by $\text{Caplet}_t^T(K)(\omega)$=`Caplet(K,i,j,k)`, again with $t =$ `i`δt, $J_t(\omega) =$`j`, and $T =$ `k`δt.

`i=k`) How to define `Caplet(K,k,j,k)` ?

`i=k-1`) As $1/(1+r_t) = Z_{t-\delta t}^t$, show that `Caplet(K,k-1,j,k)=Caplet(K,k,j,k)*Z(k-1,j,k)`.

`i<k-1`) Express `Caplet(K,i,j,k)` as a function of `Caplet(K,i+1,j,k)` and `Caplet(K,i+1,j+1,k)` when `i<k-1`, using (A.8).

`proc`) Define a procedure `Caplet(K,i,j,k)` giving the value of $\text{Caplet}_{i\delta t}^{k\delta t}(K)(\omega)$ when $J_{i\delta t}(\omega)$=`j`.

Application : In the considered Ho and Lee model (where $r_{\delta t}$=2,5%), what it the price of a contract capping at 4,5% the interests to be payed each year on a load of 1.000.000 euros during 15 years. Same question for a cap at 3,5%

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